FINANČNÍ INŽENÝRSTVÍ
KBP FFÚ & KSTP FIS

4. Numerical solution of equations of one variable

Tento učební text byl podpořen z Operačního programu Praha - Adaptabilita

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Let us have an equation

$$f(x) = 0, \tag{R}$$

where f is a function of one variable that is continuous on an interval J.

Root or **zero (point) of the function** f is such a point $c \in J$ that $f(c) = 0$.

Numerical solution of an equation:
Numerical solution of the equation (R) can be divided into two steps:
(1) Separating zeros, i.e. finding closed intervals that contain exactly one zero.
(2) Finding zeros (with given number of iterations or precision).

MATLAB - polynomials
The polynomial $p(x) = 4x^3 - 5x^2 + 2x - 1$ cane be considered as a vector $p = [4, -5, 2, -1]$.
For finding the the value of p at x use the command $y = polyval(p, x)$. If x is a vector the we get the vector with the values at particular components.

Separation of zeros, Bolzano's theorem

Bolzano's theorem: If a function f is continuous on an interval $\langle a, b\rangle$ and $f(a)f(b) < 0$, then inside the interval there is at least one zero of the equation (R).

Remark: If, moreover, the derivative of the given function is positive, resp. negative inside the interval, which means that the function is increasing on the interval or decreasing on the interval, there is only one zero in the interval.

Remark: An algebraic equation $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ has at most n real zeros. Then it suffices to find n intervals in which f does not change its sign.

Remark: If f has a continuous derivative, for the separation of zeros of (R) we evaluate values of f in endpoints of its domain and zeros of the derivative, if possible.

Example 2A: Separate roots of the algebraic equation $f(x) = 2x^3 + 3x^2 - 12x - 6 = 0$.

Find zeros of the first derivative:

$$f'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2) = 0 \quad \text{then} \quad x_1 = -2, x_2 = 1.$$

We get $f(-2) = 14, f(1) = -13$.

The equation has at most three roots because it is an algebraic equation of the 3rd degree. As $\lim\limits_{x\to-\infty} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) = \infty$, the roots are in the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$. There are exactly three roots. We can specify the intervals computing the values of f in other points: $f(-4) = -38$, $f(-3) = 3$, $f(-1) = 7$, $f(0) = -6$, $f(2) = -2$, $f(3) = 39$. Then the roots are in the intervals

$$(-4, -3),\ (-1, 0),\ (2, 3).$$

Example 2B: Separate the roots of $f(x) = x^3 + 3x^2 - 24x + 10 = 0$.

Example 2C: Separate the roots of $f(x) = x^4 - 4x + 10 = 0$.

Example 3A: Separate the zeros of $f(x) = \mathrm{e}^x - x - 2 = 0$.

Find the zeros of the first derivative:

$$f'(x) = \mathrm{e}^x - 1 = 0 \quad \text{then} \quad \mathrm{e}^x = 1 = \mathrm{e}^0 \quad \text{then} \quad x = 0.$$

We get $f(0) = \mathrm{e}^0 - 0 - 2 = -1$.

As $\lim_{x \to \pm\infty} f(x) = \infty$, the zeros are in $(-\infty, 0)$, $(0, \infty)$. In each of these interval the sign of the derivative does not change, then in each interval there is only one zero, the there are two zeros together. We specify the intervals evaluating the values of f: e.g.

$$f(-2) = \frac{1}{e^2} > 0,\ f(-1) = \frac{1}{e} - 1 < 0,\ f(1) = e - 3 < 0,\ f(2) = e^2 - 4 > 0.$$

The zeros are in the intervals with changing sign:

$$(-2, -1),\ (1, 2).$$

Example 3B: Separate zeros of $f(x) = e^x + x = 0$.

Bisection method

Let f is continuous on an interval $\langle a, b\rangle$, $f(a)f(b) < 0$ and an equation $f(x) = 0$ has only one zero c in the interval $\langle a, b\rangle$.

Bisection method:

If $f\left(\frac{a+b}{2}\right) \neq 0$, then choose such an interval out from the intervals $\left\langle a, \frac{a+b}{2}\right\rangle$, $\left\langle \frac{a+b}{2}, b\right\rangle$, whose endpoints have the opposite signs of the function values, and then bisect the chosen interval again, ...

Either we get the exact solution of the equation after n steps, or we get a sequence of intervals $\langle a_n, b_n\rangle$ with $b_n - a_n = \frac{b-a}{2^n}$.

We also get : $0 < c - a_n < \frac{b-a}{2^n}$.

Then after n steps we get either the exact zero c, or $c \approx a_{n+1} = \frac{b_n - a_n}{2}$ with error smaller than $\frac{b-a}{2^{n+1}}$.

Example 4A: Using bisection method find a root of $f(x) = 2x^3 + 3x^2 - 12x - 6 = 0$ in $(-1, 0)$, $n = 4$.

In Example 2 we determined that there is only one zero in the given interval. The values in endpoints are $f(-1) = 7 > 0,\ f(0) = -6 < 0$.
For $n = 1$ evaluate the center of the interval $\frac{-1+0}{2} = -0,5$
and the value $f(-0,5) = 0,5 > 0$.
Choose the interval, in which the sign is changing, i.e. $(-0,5; 0)$.
For $n = 2$ evaluate the center of the interval $\frac{-0,5+0}{2} = -0,25$
and the value $f(-0,25) = -2,84375 < 0$.
Choose the interval, in which the sign is changing, i.e $(-0,5; -0,25)$.
For $n = 3$ evaluate the center of the interval $\frac{-0,5-0,25}{2} = -0,375$
and the value $f(-0,375) = -1,18359375 < 0$.
Choose the interval, in which the sign is changing, i.e. $(-0,5; -0,375)$.
For $n = 4$ evaluate the center of the interval $\frac{-0,5-0,375}{2} = -0,4375$
and the value $f(-0,4375) = -0,34326171875 < 0$.
Choose the interval, in which the sign is changing, i.e. $(-0,5; -0,4375)$.

The approximate value of the zero is $c = \dfrac{-0,5 - 0,4375}{2} = -0,46875$ with error less than $\dfrac{0-(-1)}{2^5} = \dfrac{1}{32} = 0,03125$.

Example 4B: Using bisection method find a root of $f(x) = x^3 + 3x^2 - 24x + 10 = 0$ in $(0,1)$, $n = 3$.

Secant method

This method is similar to the bisection method. But we will not bisect the given interval. If $f(a) < 0$ and $f(b) > 0$, we will divide the interval in proportion $\dfrac{-f(a)}{f(b)}$, i.e.

$$x_1 = a - f(a)\frac{b-a}{f(b)-f(a)}.$$

If $f(x_1) = 0$, the process ends. In the opposite case we choose one of the intervals $\langle a, x_1\rangle$, $\langle x_1, b\rangle$, in which the signs in endpoints are different, and then repeat the process.

Using Lagrange's mean value theorem, see for instance Z.Horský, Učebnice matematiky pro posluchače VE, Věta 13,2, we can estimate error of this method.

Error estimation: $|x_n - c| \le \dfrac{|f(x_n)|}{m}$, where $m = \min\limits_{\langle a,b\rangle} |f'|$, c is the exact solution.

Example SM: Using secant method find the root of the equation $f(x) = x^3 + 3x^2 - 24x + 10 = 0$ in the interval $(0,1)$, $n = 2$.

Newton's method

If f'' is continuous on the interval $\langle a, b\rangle$, f'' and f' does not change its sign in this interval and $f(a)f(b) < 0$. Then the equation $f(x) = 0$ has exactly one solution c in the interval $\langle a, b\rangle$.

Let, moreover, $f(b)$ and $f''(b)$ have the same signs.

Let, for instance, $f(b)$ and $f''(b)$ be positive, see Fig. 4.1, function f be increasing and convex. the equation of the line tangent to the function f going through the point $[b, f(b)]$ is of the form $y = f(b) + f'(b)(x-b)$. Its x-intercept will be between a, b. We find it substituting $y = 0$ into the equation of the tangent line $0 = f(b) + f'(b)(x-b)$ then $x = b - \dfrac{f(b)}{f'(b)}$.

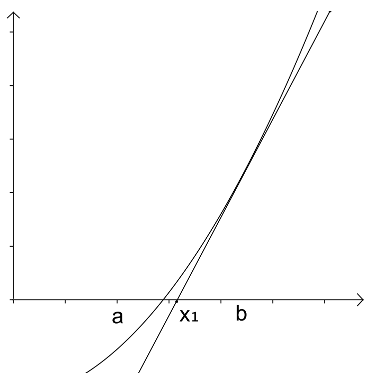

Fig. 4.1

If $f(b)$ and $f''(b)$ are negative, the function f is decreasing and concave. The x-intercept will be between a, b, as well.

Newton's method :
(1) Evaluate $x_1 = b - \dfrac{f(b)}{f'(b)}$.
(2) Evaluate $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.
...
(n) Evaluate $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$.

Using Lagrange's mean value theorem, see for instance Z.Horský, Učebnice matematiky pro posluchače VE, Věta 13,2, we can estimate error of this method.

Error estimation: $|x_n - c| \leq \dfrac{|f(x_n)|}{m}$, where $m = \min_{\langle a,b\rangle} |f'|$.

If $f(b)$ and $f''(b)$ do not have the same signs then $f(a)$ and $f''(a)$ have the same signs. We will construct the tangent line form the point $[a, f(a)]$ and in (1) evaluate $x_1 = a - \dfrac{f(a)}{f'(a)}$. The further process will be the same.

Example 5A: Using Newton's method find the root of

$$f(x) = 2x^3 + 3x^2 - 12x - 6 = 0 \text{ in the interval } \langle -4, -3\rangle, \ n = 2.$$

Evaluate values $f(-4) = -38 < 0$, $f(-3) = 3 > 0$ and the first derivative $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$, which is positive in the given interval. Moreover, it is decreasing because the second derivative is negative. The minimum of $|f'|$ in the given interval is $|f'(-3)| = f'(-3) = 24$. The second derivative $f''(x) = 12x + 6$ is negative in $\langle -4, -3\rangle$. To have the same sign as the value in the endpoint, take the endpoint -4.

For $n = 1$ evaluate
$x_1 = -4 - \dfrac{f(-4)}{f'(-4)} = -4 - \dfrac{-38}{60} \approx -3,366667$,
$f(-3,366667) \approx -7,915249$, $f'(-3,366667) \approx 35,806678$.
For $n = 2$ evaluate
$x_2 \approx -3,366667 - \dfrac{f(-3,366667)}{f'(-3,366667)} \approx -3,145612$,
$f(-3,145612) \approx -0,818899$.
The error is smaller than $\dfrac{|f(-3,145612)|}{24} = \dfrac{0,818899}{24} \approx 0,034121$.

Example 5B: Using Newton's method find the root of

$$f(x) = x^3 + 3x^2 - 24x + 10 = 0 \text{ in } \langle 0, 1\rangle, \ n = 2.$$

Combined method

We can combine the two previous methods. Let $f(b)$ and $f''(b)$ be positive, see Fig. 4.1, then the function f is increasing and convex in the given interval. Denote (s_n) the sequence of iterations obtained using secant method, (t_n) the sequence of iterations obtained using Newton's method, c is the exact solution, then for every n we get:

$$s_n < c < t_n.$$

The error is smaller than $t_n - s_n$.

Example K: Using combined method find the root of

$$f(x) = x^3 + 3x^2 - 24x + 10 = 0 \text{ in } \langle 0, 1\rangle, \; n = 2.$$

Iterative method

Let us have an equation

$$x = h(x). \tag{IR}$$

Let us have a real function of one real variable h defined in a closed interval J that maps J into J. Let there exists a constant $\alpha \in (0, 1)$ such that $|h'(x)| \le \alpha$ for all $x \in J$.
Then there is exactly one $c \in J$ such that $c = h(c)$, which means that c is the root of (IR) in J.
For an arbitrary $x_0 \in \mathbb{R}$ define the sequence (x_n) as follows (recursively): $x_{n+1} = h(x_n)$.
Then $c = \lim_{n\to\infty} x_n$ a platí $|x_n - c| \le \dfrac{\alpha^n}{1-\alpha}\,|x_0 - x_1|$.

Remark: The foregoing statement follows from Banach's fixed point theorem. The last formula determines the error of the method.

Example 6A: Using iterative method find the root of $f(x) = x^3 - 1,2x + 0,2 = 0$ in $\langle 0; 0,5\rangle$, $n = 3$.

Transform the equatin: $h(x) = \dfrac{x^3 + 0,2}{1,2} = x$.
Differentiate it: $h'(x) = \dfrac{3x^2}{1,2}$. In the given interval the function h' is positive and increasing, with maximum in $0,5$.
We get $|h'(x)| \le \dfrac{3 \cdot 0,5^2}{1,2} = 0,625 < 0,7 = \alpha < 1$ for all $x \in \langle 0; 0,5\rangle$.
We proved the assumptions of the previous Theorem. Choose

$x_0 = 0,$

$x_1 = h(0) = \dfrac{0^3 + 0,2}{1,2} \approx 0,166667,$

$x_2 = h(0,166667) = \dfrac{0,166667^3 + 0,2}{1,2} \approx 0,170525,$

$x_3 = h(0,170525) = \dfrac{0,170525^3 + 0,2}{1,2} \approx 0,170799.$

The error is smaller than $\dfrac{0,7^3}{1-0,7} \cdot |0,166667 - 0| \approx 0,190556.$

Example 6B: Using iterative method find the root of $f(x) = x^4 - 4x + 10 = 0$ in $\langle 0,1; 0,5\rangle$, $n = 2$.

Example 6C: Using iterative method find the root of $f(x) = x^3 + 10x - 15 = 0$ in $\langle 1; 1,5\rangle$, $n = 2$.

MATLAB - roots of polynomial
Use command $r = roots(p)$.
For finding zeros of a real function use command $x = fzero(fun, x_0)$, where fun is the given function, x_0 is the initial iteration.