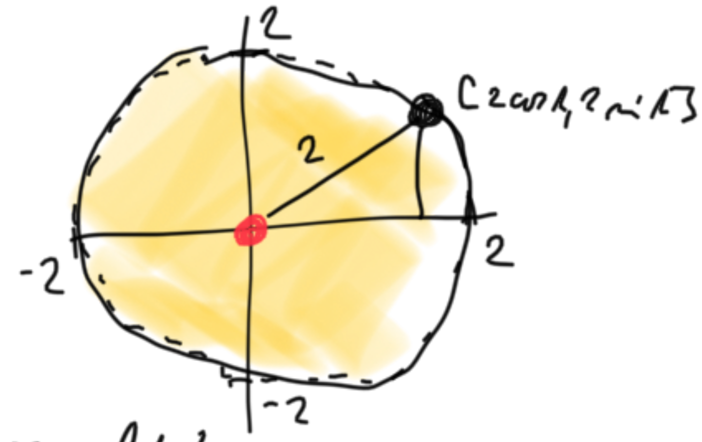
$f(x,y) = x^2 + xy$, $M = \{[x,y] \in \mathbb{R}^2;\ x^2+y^2 < 4\}$



f spojitá $\mathbb{R}^2$, M je omezená

M není uzavřená (je otevřená), tedy není kompaktní

$\nabla f(x,y) = [2x+y, x] = o \iff x=0 \ \& \ y=0$

$f(0,0)=0$

Budu studovat f na $\overline{M}$, což je kompaktní množina, tedy na $\overline{M}$ nabývá f max. i min.

$\overline{M} = M \cup H(M)$.

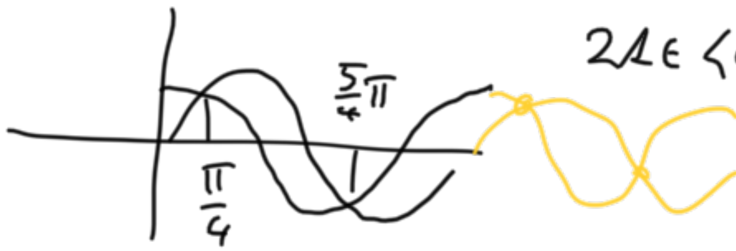
Hranice: $H(M) = \{x^2+y^2=4\}$

$x = 2\cos t$
$y = 2\sin t$, $t \in \langle 0, 2\pi \rangle$

$g(t) = f(2\cos t, 2\sin t) = 4\cos^2 t + 4\sin t\cos t$, $t \in \langle 0, 2\pi)$

$g'(t) = 8\cos t(-\sin t) + 4\cos^2 t - 4\sin^2 t = 4(\cos 2t - \sin 2t)$

$= 0 \iff \sin 2t = \cos 2t$

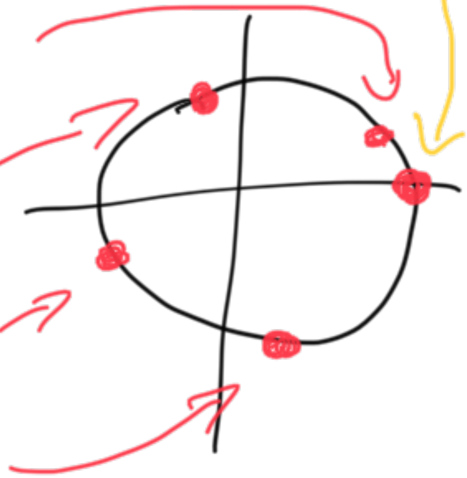


$2t \in \langle 0, 4\pi \rangle$

$2t = \frac{\pi}{4},\ \frac{5}{4}\pi,\ \frac{9}{4}\pi,\ \frac{13}{4}\pi$

P.B.:

$t = \frac{\pi}{8},\ \frac{5}{8}\pi,\ \frac{9}{8}\pi,\ \frac{13}{8}\pi$



Trik: $g(t) = 2 \cdot 2\sin t\cos t + 4\cos^2 t - 2\sin^2 t + 2\sin^2 t =$

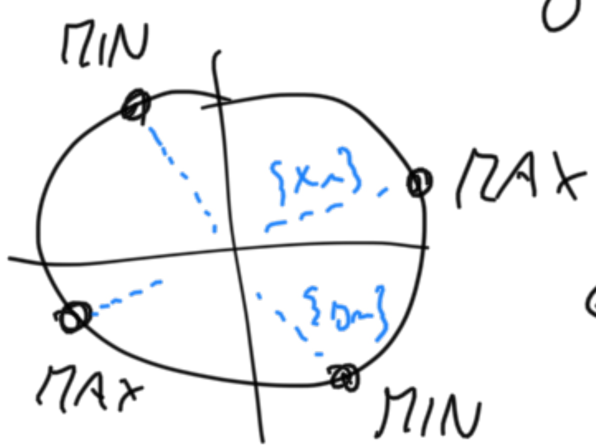
$= 2\sin 2t + 2\cos 2t + 2 = 2(\sin 2t + \cos 2t + 1)$

dosazení

$g(0) = 4$

$g\left(\frac{\pi}{8}\right) = g\left(\frac{9}{8}\pi\right) = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1\right) = 2 + 2\sqrt{2}$ MAX

$g\left(\frac{5}{8}\pi\right) = g\left(\frac{13}{8}\pi\right) = 2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1\right) = 2 - 2\sqrt{2}$ MIN



$\in M$

Heine: $\left.\begin{array}{l} f(x_n) \to 2+2\sqrt{2} \\ f(y_n) \to 2-2\sqrt{2} \end{array}\right\} \Rightarrow \sup_M f = 2+2\sqrt{2}$

$\inf_M f = 2-2\sqrt{2}$!

ale těchto hodnot se na M nenabývá

$$f(x,y) = \frac{x-y}{x^2+y^2+1}, \quad M = \mathbb{R}^2$$

M není omezená

f je spojitá na $\mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^2+y^2+1) - (x-y)2x}{(x^2+y^2+1)^2} = \frac{-x^2+2xy+y^2+1}{(x^2+y^2+1)^2} = 0$$

$$\Leftrightarrow$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-(x^2+y^2+1) - (x-y)2y}{(x^2+y^2+1)^2} = \frac{-x^2-2xy+y^2-1}{(x^2+y^2+1)^2} = 0$$

$$-x^2+2xy+y^2+1 = 0$$
$$-x^2-2xy+y^2-1 = 0$$

sečteme: $-2x^2+2y^2 = 0$

$$x^2 = y^2$$

$$y = \pm x$$

dosadím do 1. rovnice:

$y=x$: $\quad -x^2+2x^2+x^2+1=0$

$$2x^2+1=0$$

$$\times$$

$y=-x$: $\quad -x^2-2x^2+x^2+1=0$

$$2x^2=1$$

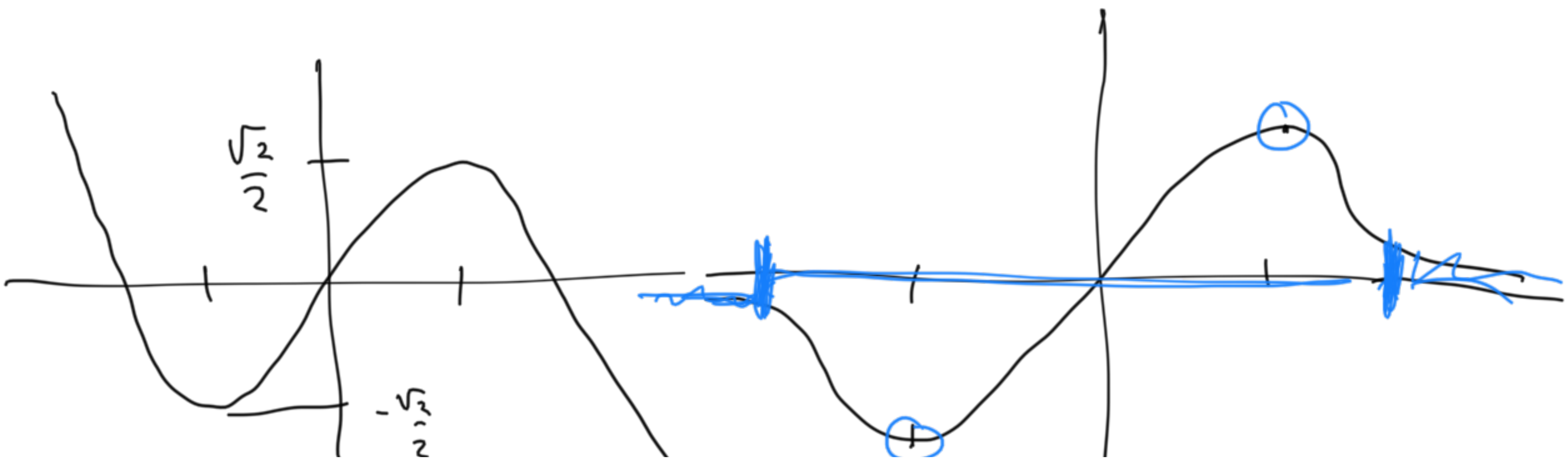
$$x=\pm\frac{\sqrt{2}}{2}$$

P.B.

$$\boxed{\begin{array}{c}\left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right] \\ \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]\end{array}}$$

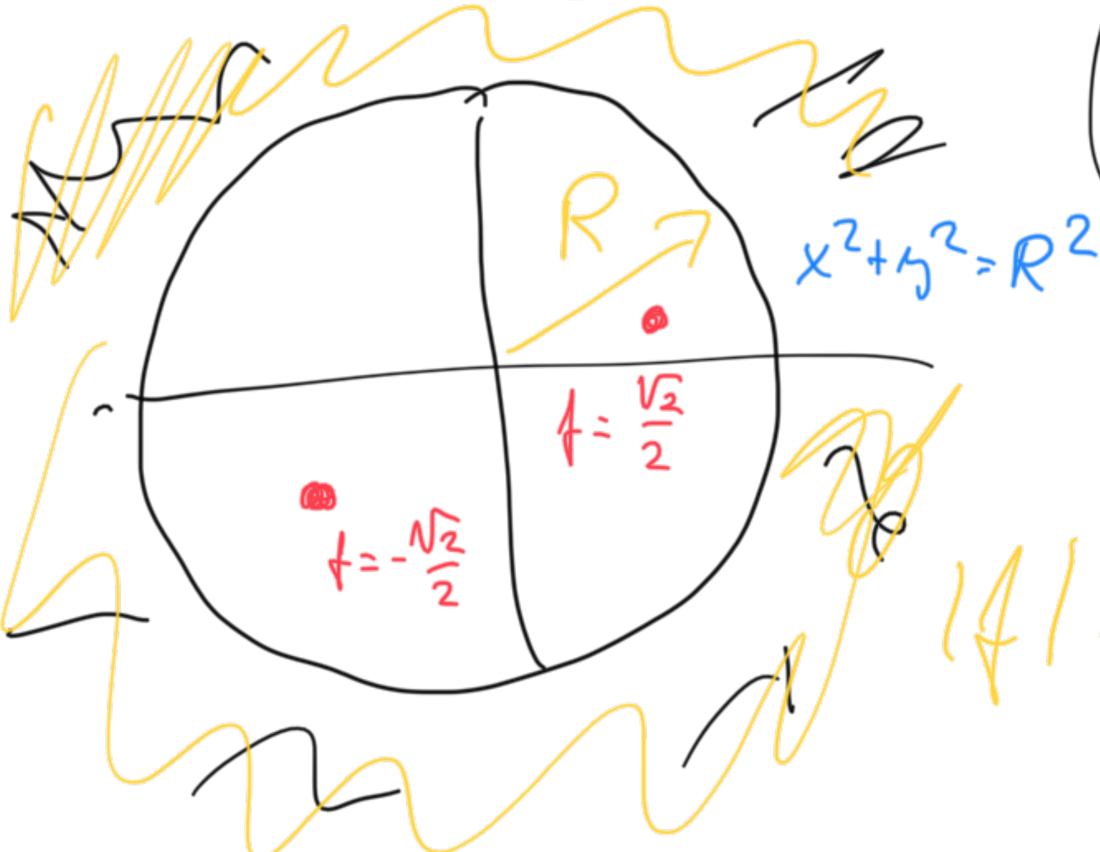
$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2}$$

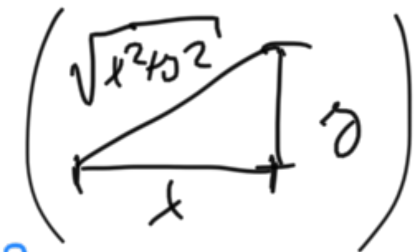
$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{2}}{2}$$



$$|f(x,y)| \underset{\uparrow}{\leq} \frac{|x|}{x^2+y^2+1} + \frac{|y|}{x^2+y^2+1} \leq \frac{\sqrt{x^2+y^2}}{x^2+y^2+1} + \frac{\sqrt{x^2+y^2}}{x^2+y^2+1} \leq 2\,\frac{\sqrt{x^2+y^2}}{x^2+y^2} =$$

Δ nerovnost





$$= \frac{2}{\underbrace{\sqrt{x^2+y^2}}_{R}}$$

$$|f| \leq \frac{2}{R}$$

Stačí vzít $R = 4$

$$\boxed{\text{Mimo kouli } B(0,4) \text{ je } |f| \leq \frac{2}{4} = \frac{1}{2} < \frac{\sqrt{2}}{2}}$$

Uz. koule $\{x^2+y^2 \leq 4\}$ je kompaktní, na ní nabývá f max. i min. a to v $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right\}$, $MAX = \frac{\sqrt{2}}{2}$, $MIN = -\frac{\sqrt{2}}{2}$)

... že $-\frac{\sqrt{2}}{2} < f < \frac{\sqrt{2}}{2}$.

Závěr: f na $\mathbb{R}^2$ nabývá max. i min.

Díky tomuto to nemusíme řešit na kraji $\{x^2+y^2=4\}$, neboť tam je $|f| < \frac{\sqrt{2}}{2}$.

$f(x,y) = 3x^2 + 4y^3$, $\Pi = \{x^2+y^2=1\} = \{g(x,y)=0\}$

$g(x,y) = x^2+y^2-1$

$G = \mathbb{R}^2$, $f, g \in C^1(G)$ (dokonce C^∞ ... polynomy)

Bod $[x,y]$ podezřelý z extrému splňuje

buď (I): $\nabla g(x,y) = 0$

nebo (II): $\exists \lambda \in \mathbb{R}$: $\nabla f(x,y) + \lambda \nabla g(x,y) = 0$

$\nabla g(x,y) = [2x, 2y]$

$\nabla f(x,y) = [6x, 12y^2]$

$$\frac{\partial f}{\partial x}(x,y) + \lambda \frac{\partial g}{\partial x}(x,y) = 0$$

$$\frac{\partial f}{\partial y}(x,y) + \lambda \frac{\partial g}{\partial y}(x,y) = 0$$

(I) $\nabla g(x,y) = o \iff x=0,\ y=0 \notin M$, tedy to není bod podezřelý z extrému

(II)

$$\boxed{\begin{aligned} 6x + \lambda \cdot 2x &= 0 \quad &| \cdot y \\ 12y^2 + \lambda \cdot 2y &= 0 \quad &| \cdot x \\ x^2 + y^2 &= 1 \end{aligned}}$$

} odečteme

(je třeba, aby $[x,y] \in M$)

Soustava 3 rovnic o 3 neznámých, ale nezajímá mě, kolik vyjde λ.

$$6xy + 2\cancel{\lambda xy} - 12y^2x - 2\cancel{\lambda xy} = 0$$

$$xy - 2xy^2 = 0 \quad \begin{cases} x = 0 \\ y = 2y^2 \quad \begin{cases} y = 0 \\ y = \frac{1}{2} \end{cases} \end{cases}$$

$$x^2 + y^2 = 1$$

P.B.

$[0, \pm 1]$

$[\pm 1, 0]$

$[\pm \frac{\sqrt{3}}{2}, \frac{1}{2}]$