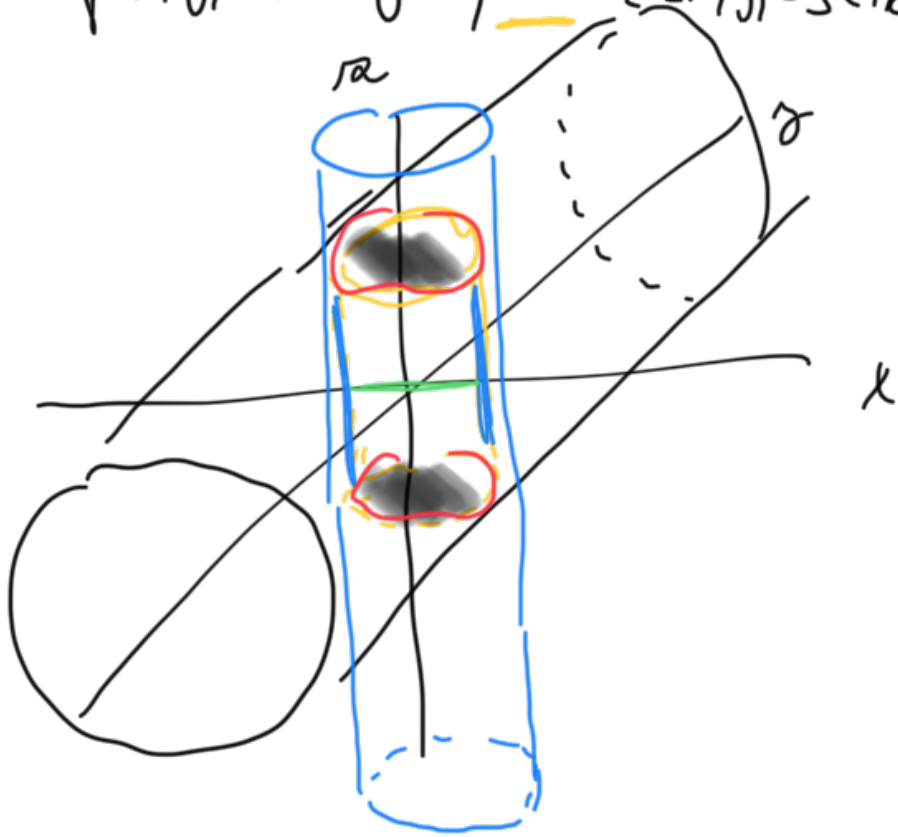
$f(x,y,z) = yz$, $M = \{[x,y,z] \in \mathbb{R}^3;\ \underbrace{x^2+z^2 \le 4}_{g_2},\ \underbrace{x^2+y^2 \le 1}_{g_1}\}$



M je omezená (např. $|z| \le 2$, $|x| \le 1$, $|y| \le 1$)
M je uzav. (průnik uzavřených)
} Kompakt.

f je spojitá na $\mathbb{R}^3$

f nabývá extrémů na M.

$f, g_1, g_2 \in C^1(\mathbb{R}^3)$ (Lagr. věta)

$M = \operatorname{Int} M \cup H(M) = \operatorname{Int} M \cup M_1 \cup M_2 \cup M_3$

$\operatorname{Int} M = \{x^2+z^2 < 4,\ x^2+y^2 < 1\}$

$M_1 = \{x^2+z^2 < 4,\ x^2+y^2 = 1\}$

$M_2 = \{x^2+z^2 = 4,\ x^2+y^2 < 1\}$

$M_3 = \{x^2+z^2 = 4,\ x^2+y^2 = 1\}$

$g_1(x,y,z) = x^2+y^2-1$

$g_2(x,y,z) = x^2+z^2-4$

na Int M:

$\nabla f(x,y,z) = [0, z, y] = 0 \iff y = z = 0$

P.B. $[x,0,0]$, $x \in (-1,1)$

$f(x,0,0)=0$

Na M_1: $\nabla g_1(x,y,z) = [2x, 2y, 0] = o \iff x=y=0 \notin M_1$

$\exists \lambda \in \mathbb{R}: \quad \nabla f + \lambda \nabla g_1 = o$

$$\begin{aligned} 0 + \lambda \cdot 2x &= 0 \\ z + \lambda \cdot 2y &= 0 \\ y &= 0 \end{aligned}$$

Položím $\lambda' = 2\lambda$

$$\begin{aligned} 0 + \lambda' \cdot x &= 0 \\ z + \lambda' \cdot y &= 0 \\ y &= 0 \end{aligned}$$

Tedy stačí řešit soustavu

$$\left.\begin{aligned} 0 + \lambda x &= 0 \\ z + \lambda y &= 0 \\ y &= 0 \\ \hline x^2+y^2 &= 1 \end{aligned}\right. \quad z = 0, \quad \Rightarrow x = \pm 1$$

P.B. $[\pm 1, 0, 0] \in M_1$

$(1^2 + 0^2 < 4)$

$f(\pm 1, 0, 0) = 0$

Na M_2: $\nabla g_2(x,y,z) = [2x, 0, 2z] = o \iff x=z=0 \notin M_2$

$\nabla f + \lambda \nabla g_2 = o$

$$\begin{aligned} 0 + \lambda \cancel{2}x &= 0 \\ z + \lambda \cdot 0 &= 0 \\ y + \lambda \cancel{2}z &= 0 \end{aligned}$$

$$\lambda x = 0$$
$$z = 0 \qquad y = 0$$
$$y + \lambda z = 0$$
$$x^2 + z^2 = 4$$

$$x = \pm 2$$

$$[\pm 2, 0, 0] \notin M_2$$
$$(2^2 + 0^2 \not\leq 1)$$

Na M_3: $\nabla g_1, \nabla g_2 < 2$:

$\nabla g_1 = [2x, 2y, 0]$, $\nabla g_2 = [2x, 0, 2z]$

- $\nabla g_2 = \vec{0} \iff x = z = 0 \notin M_3$
- $\nabla g_1 = c \cdot \nabla g_2$

$$\begin{aligned} 2x &= c \cdot 2x \\ 2y &= c \cdot 0 \quad \Rightarrow y = 0 \\ 0 &= c \cdot 2z \quad \Rightarrow \begin{cases} z = 0 \\ c = 0 \Rightarrow x = 0 \end{cases} \end{aligned}$$

$x^2 + y^2 = 1$

$y = z = 0$, $x = \pm 1$
&
$x = \pm 2$

$\nabla f + \lambda \nabla g_1 + \mu \nabla g_2 = 0$:

$$0 + \lambda 2x + \mu 2x = 0$$
$$z + \lambda 2y = 0$$
$$y + \mu 2z = 0$$

$$\left.\begin{aligned} \lambda x + \mu x &= 0 \quad /\cdot z \\ z + \lambda y &= 0 \\ y + \mu z &= 0 \quad /\cdot x \end{aligned}\right\} \text{odečtu}$$

$$x^2 + y^2 = 1$$
$$x^2 + z^2 = 4$$

$$\lambda x z - x y = 0 \quad /\cdot y$$
$$\lambda y + z = 0 \quad /\cdot xz$$

$$x y^2 + x z^2 = 0$$
$$x(y^2+z^2) = 0$$

$x = 0 \rightarrow y = \pm 1, \; z = \pm 2$

$y = z = 0 \rightarrow x = \pm 1, \; x = \pm 2$ ×

P.B. $[0, \pm 1, \pm 2]$

$f(0,1,2) = f(0,-1,-2) = 2$

MAX

$f(0,-1,2) = f(0,1,-2) = -2$

MIN

$$
\begin{pmatrix} -3 & 2 & 1 & 2 & -8 \\ 1 & 9 & 3 & 7 & 1 \\ 10 & 11 & 10 & 9 & 20 \\ 9 & 2 & 7 & 2 & 19 \\ 4 & 0 & 2 & 0 & 9 \end{pmatrix}
\sim
\begin{array}{r} \mathrm{I.}+\mathrm{V.} \\ \\ \mathrm{III.}-\mathrm{IV.} \\ \mathrm{IV.}+3\cdot\mathrm{I.} \\ \\ \end{array}
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 9 & 3 & 7 & 1 \\ 1 & 9 & 3 & 7 & 1 \\ 0 & 8 & 10 & 8 & -5 \\ 4 & 0 & 2 & 0 & 9 \end{pmatrix}
\sim
\begin{array}{r} \\ -\mathrm{I.} \\ -\mathrm{II.} \\ \\ -4\cdot\mathrm{I.} \end{array}
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 7 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 10 & 8 & -5 \\ 0 & -8 & -10 & -8 & 5 \end{pmatrix}
\sim
$$

$$
\sim
\begin{array}{r} \\ \\ \\ \\ +\mathrm{IV.} \end{array}
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 7 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 10 & 8 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\sim
\begin{array}{r} \\ \\ \\ -\mathrm{II.} \\ \\ \end{array}
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 7 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\sim
$$

$$
\sim
\begin{array}{r} \\ -7\cdot\mathrm{IV} \\ \\ \\ \\ \end{array}
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & -70 & -16 & 35 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\sim
\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 10 & 3 & -5 \\ 0 & 0 & -70 & -16 & 35 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

výměna
II. a IV.
IV. a III.

$\Rightarrow h(A) = 3$

$$
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & x & x+1 \\ y & y+1 & 15 & 16 \end{pmatrix}
\sim
\begin{array}{cccc} \mathrm{II}.-\mathrm{I}. & & & \mathrm{IV}.-\mathrm{III}. \\ \end{array}
\begin{pmatrix} 1 & 1 & 3 & 1 \\ 5 & 1 & 7 & 1 \\ 9 & 1 & x & 1 \\ y & 1 & 15 & 1 \end{pmatrix}
\sim
\begin{array}{c} \begin{array}{cccc} \mathrm{II}.-\mathrm{IV}. & \mathrm{IV}. & \mathrm{I}. & \mathrm{III}. \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 9 & x \\ 0 & 1 & y & 15 \end{pmatrix} \end{array}
$$

$$
\sim
\begin{array}{c} \\ -\mathrm{I}. \\ -\mathrm{I}. \\ -\mathrm{I}. \end{array}
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 8 & x-3 \\ 0 & 0 & y-1 & 12 \end{pmatrix}
\sim
\begin{array}{c} \\ \cdot\frac{1}{4} \\ \\ \\ \end{array}
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 8 & x-3 \\ 0 & 0 & y-1 & 12 \end{pmatrix}
\sim
\begin{array}{c} \\ \\ -8\cdot\mathrm{II} \\ \\ \end{array}
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x-11 \\ 0 & 0 & y-1 & 12 \end{pmatrix}
$$

$y=1$ $\sim$

$$
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x-11 \\ 0 & 0 & 0 & 12 \end{pmatrix}
\sim
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{array}{l} \\ \\ \mathrm{IV}. \\ \mathrm{III}. - \frac{(x-11)}{12}\cdot \mathrm{IV}. \end{array}
\qquad h(A)=3
$$

$y \neq 1$ $\sim$

$$
\begin{array}{c} \\ \\ \\ -(y-1)\cdot\mathrm{II}. \end{array}
\begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x-11 \\ 0 & 0 & 0 & 13-y \end{pmatrix}
$$

$y=13, x\neq 11$ .. $h(A)=3$

$x=11, y\neq 13$ ... $h(A)=3$

$x=11, y=13$ ... $h(A)=2$

$\dots\ h(A) = 2$

$x \neq 11,\ y \neq 13\ \dots\ h(A) = 3$

$$\sim \begin{pmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x-11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \overline{IV} - \frac{(13-y)}{(x-11)} \cdot \overline{III}$$

Dokonaly:

$x = 11\ \&\ y = 13 \Rightarrow h(A) = 2$

jinak $\Rightarrow h(A) = 3$