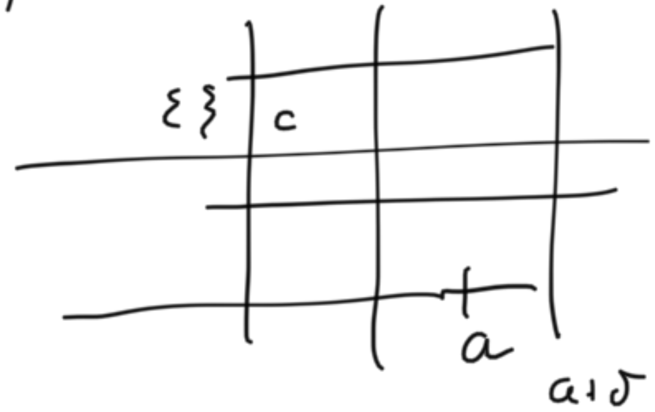
$c \in \mathbb{R}$, $f(x) = c$, $x \in \mathbb{R}$ konstantní

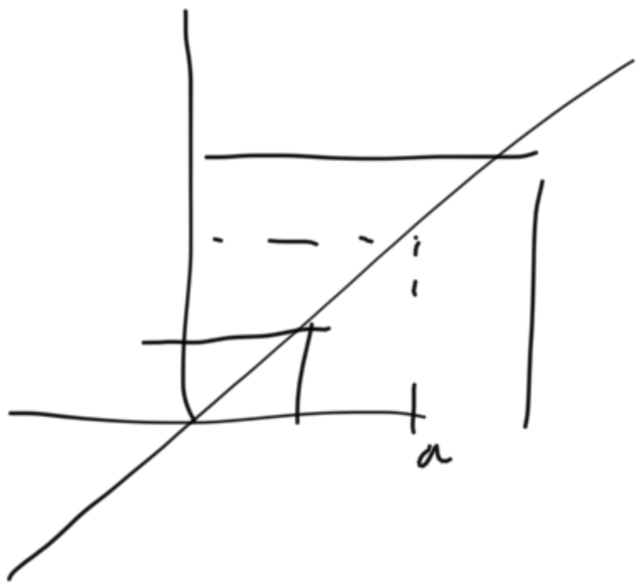


Pro $\varepsilon > 0$ lze volit $\delta > 0$ libovolné a vždy platí, že $\forall x \in P(a,\delta) : f(x) = c \in B(c,\varepsilon)$.

Tedy $\forall a \in \mathbb{R}^* : \lim\limits_{x \to a} f(x) = c \underset{a \in \mathbb{R}}{=} f(a)$

Tedy f je spojitá v každém bodě $\mathbb{R}$.

$f(x) = x$, $x \in \mathbb{R}$



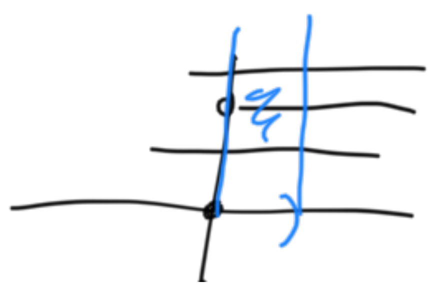
Pro $\varepsilon > 0$ vezmeme $\delta = \varepsilon$.

Pak $\forall x \in P(a,\delta) : f(x) = x \in P(a,\overset{\varepsilon}{\overset{\shortparallel}{\delta}}) \subset B(a,\varepsilon)$

Tedy $\forall a \in \mathbb{R}^* : \lim\limits_{x \to a} f(x) = a \underset{a \in \mathbb{R}}{=} f(a)$

Tedy f je spojitá v každém bodě $\mathbb{R}$.

$$\operatorname{sgn} x = \begin{cases} 1 & \ldots\ x > 0 \\ 0 & \ldots\ x = 0 \\ -1 & \ldots\ x < 0 \end{cases}$$



$\lim_{x\to 0+} \operatorname{sgn} x = 1 \neq 0 = f(0)$ Pro $\varepsilon > 0$ lze vzít libovolné $\delta > 0$.

Pak $\forall x \in P^+(0,\delta) : f(x) = 1 \in B(1,\varepsilon)$.

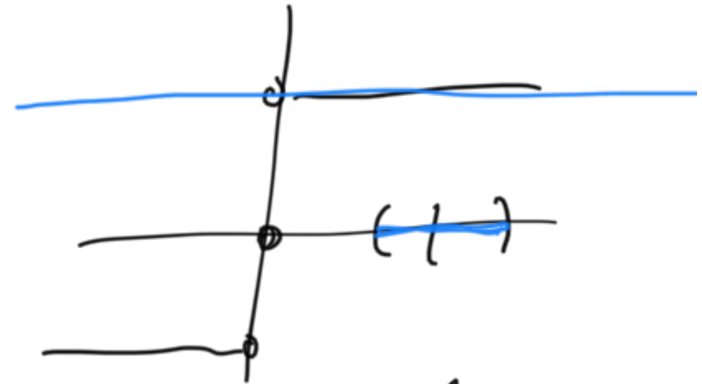
$P^+(0,\delta) = (0,\delta)$

Podobně $\lim_{x\to 0-} \operatorname{sgn} x = -1 \neq 0 = f(0)$

$\lim_{x\to 0} \operatorname{sgn} x$ neexistuje

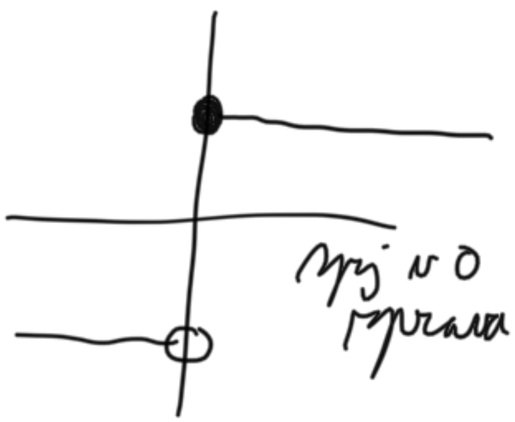
Tedy $\operatorname{sgn} x$ není spojitá v 0.

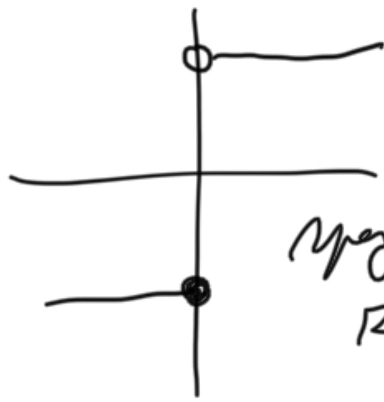
sgn x není v 0 spojitá ani zprava, ani zleva



$$\lim_{x\to a} \operatorname{sgn} x = \begin{cases} 1, & a > 0 \\ -1, & a < 0 \end{cases}$$

Tedy $\operatorname{sgn} x$ je spojitá v $\forall a \in \mathbb{R} \setminus \{0\}$.





f je spojitá v $c \in \mathbb{R} \iff f$ je spojitá v c zleva & f je spojitá v c zprava

<u>Důk. V20</u>: Nechť $\lim f(x) = A \in \mathbb{R}$. Vezmeme $\varepsilon = 1$.

Pak $\exists \delta > 0 : \forall x \in P(c,\delta) : f(x) \in B(A,1)$

$A-1 < f(x) < A+1$

$\square$

Důk. V21: (i) $A \in \mathbb{R}$, $B = +\infty$

$A+B = +\infty$, tj. CHCI $\lim_{x \to c} (f(x)+g(x)) = +\infty$

Nechť $K \in \mathbb{R}$. CHCI: $\exists \delta > 0 : \forall x \in P(c,\delta) : f(x)+g(x) > K$

VIM: $\exists \delta_1 > 0 : \forall x \in P(c,\delta_1) : A-1 < f(x) < A+1$ $(\varepsilon = 1)$

$\exists \delta_2 > 0 : \forall x \in P(c,\delta_2) : g(x) > L = K-A+1$

Pak pro $\delta = \min\{\delta_1, \delta_2\}$ platí, že

$\forall x \in P(c,\delta) \subset P(c,\delta_1), \subset P(c,\delta_2) : f(x) > A-1 \ \& \ g(x) > L \overset{K-A+1}{=}$

$\Downarrow$

$f(x)+g(x) > A-1+L = K$

(iii) $A, B \in \mathbb{R}$, $B \neq 0$ ($\frac{A}{B}$ je definován)

$\lim_{x \to c} g(x) = B \neq 0 : \exists \eta > 0 : \forall x \in P(c,\eta) : |g(x)-B| < \frac{|B|}{2}$ > 0

$|B| - |g(x)| \le \big||B| - |g(x)|\big|$ ← △ nerovnost

tj. $\boxed{|g(x)| > \frac{|B|}{2}}$

speciálně $g(x) \neq 0$, tedy $\frac{f}{g}(x)$ je definováno pro $x \in P(c,\eta)$

Mějme $\varepsilon > 0$.

$$\exists \delta_1 > 0 : \forall x \in P(c,\delta_1) : |f(x) - A| < \frac{\varepsilon}{4} \cdot |B|$$

$$\exists \delta_2 > 0 : \forall x \in P(c,\delta_2) : |g(x) - B| < \frac{\varepsilon}{4} \cdot \frac{|B|^2}{|A|}$$

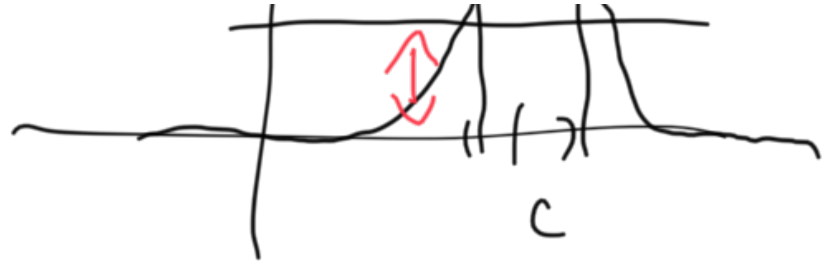
Položme $\delta = \min\{\eta, \delta_1, \delta_2\}$. Pak $\forall x \in P(c,\delta)$:

$$\left| \frac{f(x)}{g(x)} - \frac{A}{B} \right| = \frac{|f(x) \cdot B - A \cdot g(x)|}{|g(x) \cdot B|} = \frac{|f(x) \cdot B - A \cdot B + A \cdot B - A g(x)|}{|g(x)| \cdot |B|} =$$

$$= \frac{|B(f(x) - A) + A(B - g(x))|}{|g(x)| \cdot |B|} \underset{\triangle \text{ nerovnost}}{\le} \frac{|B| \cdot |f(x) - A| + |A| \cdot |B - g(x)|}{|g(x)| \cdot |B|} =$$

$$= \underbrace{\frac{1}{|g(x)|}}_{\text{potřebuji, aby nebylo moc velké}} \cdot \left(|f(x) - A| + \frac{|A|}{|B|} \cdot |g(x) - B| \right) \le \frac{2}{|B|} \cdot \left(|f(x) - A| + \frac{|A|}{|B|} |g(x) - B| \right) =$$

$$= \underbrace{\frac{2}{|B|} \cdot |f(x) - A|}_{< \frac{\varepsilon}{2}} + \underbrace{\frac{2|A|}{|B|^2} \cdot |g(x) - B|}_{< \frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$



$\frac{\varepsilon}{2}$ $\quad < \frac{\varepsilon}{2}$ $\quad \square$

Pozn.: Ukázali jsme si, že je-li $\lim\limits_{x\to c} g(x) \neq 0$, pak

$$\exists \eta > 0\ \forall x \in P(c,\eta) : g(x) \neq 0.$$

Např. spojitost součinu:

f, g jsou spojité v c

$$\lim_{x\to c} (f\cdot g)(x) \overset{AL}{=} \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) \overset{\Uparrow}{=} \underbrace{\underset{\in \mathbb{R}}{f(c)} \cdot \underset{\in \mathbb{R}}{g(c)}} = (f\cdot g)(c)$$

Tedy $f\cdot g$ je spojitá v c.

Důk.: a) $A \in \mathbb{R},\ A > 0$ jedna lomeno „kladná nula“

Nechť $L \in \mathbb{R}$ libovolné.

$$\exists \delta_1 > 0 : \forall x \in P(c,\delta_1) : \underset{= \frac{A}{2}}{A - \frac{A}{2}} < f(x) < A + \frac{A}{2}$$

$$\exists \delta_2 > 0 : \forall x \in P(c,\delta_2) : |g(x) - 0| < \frac{A}{2(|L|+1)}$$

$\|$

$|g(x)|$

Položme $\delta = \min\{\eta, \delta_1, \delta_2\}$. Pak $\forall x \in P(c,\delta)$:

$$0 < g(x) < \frac{A}{2(|L|+1)}, \quad \text{tedy}$$

$$\frac{f(x)}{g(x)} > \frac{A/2}{\frac{A}{2(|L|+1)}} = |L|+1 > L$$

b) $A = +\infty$

Zvolme $L \in \mathbb{R}$ libovolné.

$$\exists \delta_1 > 0 \ \forall x \in P(c,\delta_1): \ f(x) > 1$$

$$\exists \delta_2 > 0 \ \forall x \in P(c,\delta_2): \quad |g(x)| < \frac{1}{|L|+1}$$

Položme $\delta = \min\{\eta, \delta_1, \delta_2\}$. Pak $\forall x \in P(c,\delta)$:

$$0 < g(x) < \frac{1}{|L|+1}, \quad \text{tedy}$$

$$\frac{f(x)}{g(x)} > \frac{1}{\frac{1}{|L|+1}} = |L|+1 > L$$

$\square$

Pozn.: Platí i varianty pro jednostranné limity, např.

$$\lim_{x\to c+} \big(f(x)+g(x)\big) = \lim_{x\to c+} f(x) + \lim_{x\to c+} g(x),$$

je-li pravá strana definována.