- Dle V2G $\lim_{x\to+\infty} \log x$ existuje (log rostoucí (L2))

$$\| $$

$$\sup\{\log x,\ x\in(0,+\infty)\} = +\infty$$

$$\log 2^n = \underbrace{n\cdot\log 2}_{\downarrow\ +\infty},\quad \log 2 > 0 \quad (\log 1 = 0,\ \log \text{ je rostoucí})$$

$\lim_{x\to 0+} \log x$ existuje

$$\| $$

$$\inf\{\log x,\ x\in(0,+\infty)\} = -\infty$$

$$\log 2^{-n} = -n\cdot\log 2 \to -\infty$$

- log je spojitá fce:

a) log je spojitý v 1: na $P(1,\delta)$ (dokonce všude mimo 1).

$$\lim_{x\to 1} \log x = \lim_{x\to 1} \frac{\log x}{x-1}\cdot(x-1) \overset{AL}{=} \lim_{x\to 1}\frac{\log x}{x-1}\cdot\lim_{x\to 1}(x-1) =$$

$$\overset{(L4)}{=} 1\cdot 0 = 0 = \log 1$$

b) Nechť $c \in (0,+\infty)$.

log je spojitý v c:

$$\lim_{x\to c}\log x = \lim_{x\to c}\log\left(\frac{x}{c}\cdot c\right) \overset{(L3)}{=}$$

$$= \lim_{x\to c}\left(\log\frac{x}{c}+\log c\right) \overset{AL}{\underset{?}{=}} \lim_{x\to c}\log\frac{x}{c} + \overbrace{\lim_{x\to c}\log c}^{\log c} =$$

$$= 0 + \log c = \log c$$

LIM. SLOŽ. FCE:

$f(y) = \log y$ | $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c}\frac{x}{c} \overset{AL}{=} 1 \ (=A)$

$g(x) = \frac{x}{c}$ | $\lim\limits_{y\to A} f(y) = \lim\limits_{y\to 1}\log y \overset{a)}{=} \log 1 = 0$

Jsou splněny podm. (S) i (P)

$\uparrow$ a) $\quad$ $\uparrow$ g je prostá ($c \neq 0$)

$$\Rightarrow \lim \log(x) = 0$$

$\lim_{x\to c}$ ... 0

$\parallel$

$$\lim_{x\to c}\log\frac{x}{c}$$

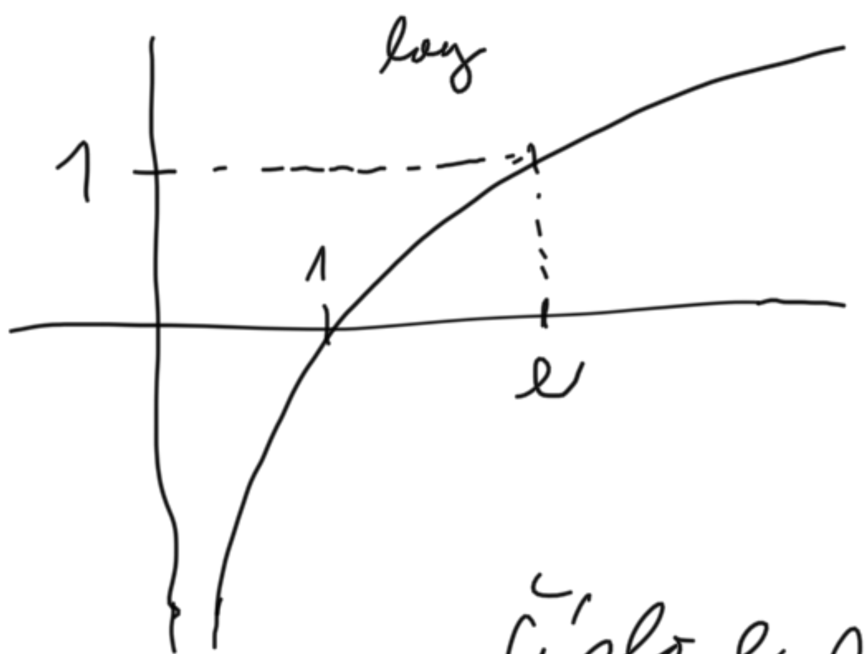
- $H_{\log}=\mathbb{R}$:

log je spojitá

$H_{\log}=\log((0,+\infty))$ ... je interval (V30)

$\lim_{x\to+\infty}\log x=+\infty$, $\lim_{x\to 0+}\log x=-\infty$ $\Rightarrow$ log není omez. shora ani zdola

$\Rightarrow H_{\log}=\mathbb{R}$



- $\exists!\, e\in(0,+\infty)$: $\log e=1$

$\exists$ plyne z $H_{\log}=\mathbb{R}$

jednoz. plyne z prostoty log (L2)

Číslo e je iracionální, $e \doteq 2{,}71828$

- $\exp(x+y)=(\exp x)\cdot(\exp y)$

(L3)

$$\log\big((\exp x)\cdot(\exp y)\big) = \log(\exp x) + \log(\exp y) =$$
$$= x + y \qquad / \exp$$

$$\cancel{\exp}\Big(\cancel{\log}\big((\exp x)\cdot(\exp y)\big)\Big) = \exp(x+y)$$
$$\parallel$$
$$(\exp x)\cdot(\exp y)$$

- $(\exp x)\cdot\exp(-x) = \exp(x+(-x)) = \exp 0 = 1$

$$\Rightarrow \exp(-x) = \frac{1}{\exp x} \qquad (\exp x \neq 0)$$

- $\exp(nx) = (\exp x)^n$ indukcí / nebo:

$$\log\big((\exp x)^n\big) = n\log(\exp x) =$$
$$= n\cdot x \qquad / \exp$$
$$(\exp x)^n = \exp\big(\log(\exp x)^n\big) = \exp(n\cdot x)$$

- $\lim\limits_{x\to+\infty} \exp x \overset{V26}{=} \sup H_{\exp} = \sup(0,+\infty) = +\infty$

$\lim \exp x \overset{V26}{=} \inf H_{\exp} = \inf(0,+\infty) = 0$

$$\lim_{x\to -\infty} \exp x = \inf \exp = \inf \exp(0,+\infty) = 0$$

- $$\lim_{x\to 0} \frac{\exp x - 1}{x} = \lim_{x\to 0} \frac{1}{f\circ g(x)} \overset{AL}{=} \frac{1}{\lim_{x\to 0} f\circ g(x)} = \frac{1}{1} = 1$$

LIM. SL. FCE: $f(y) = \dfrac{\log y}{y-1}$

$g(x) = \exp x$

$$f\circ g(x) = \frac{\log(\exp x)}{\exp x - 1} = \frac{x}{\exp x - 1}$$

exp je spojitá

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \exp x \overset{\downarrow}{=} \exp 0 = 1$$

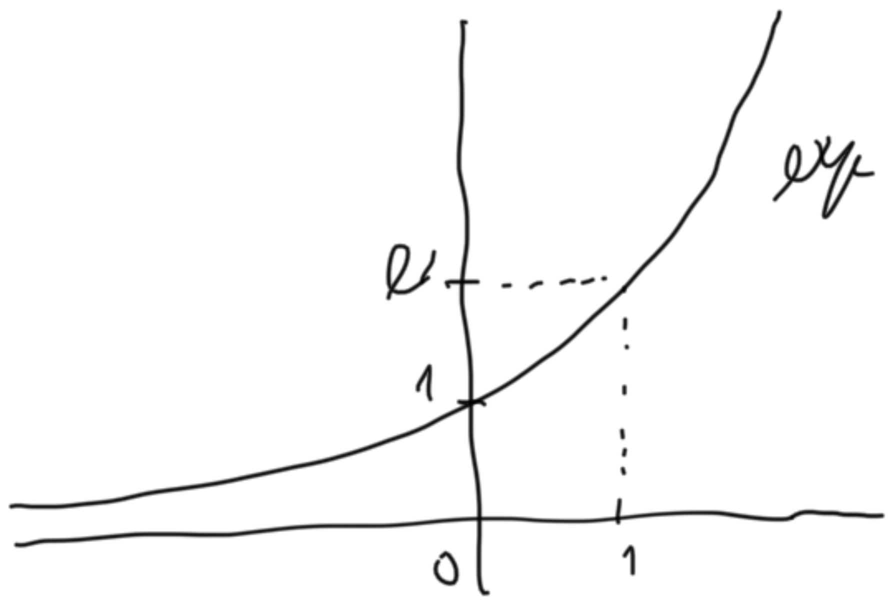
$$\lim_{y\to 1} f(y) \overset{(L4)}{=} 1$$

POZOR: není splněna podm. (S)

f není v bodě 1 ani DEFINOVANÁ

(P) je splněna (exp je rostoucí)

$$\Rightarrow \lim_{x\to 0} f\circ g(x) = 1$$



• $r = \frac{p}{q}$, $e^r = \sqrt[q]{e^p} = \exp(r)$

$\hookrightarrow$ visto $q \in \mathbb{N}$, $x \in \mathbb{R}$: $\quad \left(\exp \frac{x}{q}\right)^q = \exp\left(q \cdot \frac{x}{q}\right) = \exp x > 0$

$$\Rightarrow \exp \frac{x}{q} = \sqrt[q]{\exp x}$$

$r \in \mathbb{Q}$, vale $r = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$

$$\exp r = \exp \frac{p}{q} = \sqrt[q]{\exp p} = \sqrt[q]{\exp(p \cdot 1)} = \sqrt[q]{(\exp 1)^p} =$$
$$= \sqrt[q]{e^p}$$

$\forall x \in \mathbb{R}$:

$$e^x = \exp(x \cdot \overbrace{\log e}^{1}) = \exp x \qquad \text{ok}$$

b / $a \in \mathbb{R}$, $a > 0$, $b \in \mathbb{R}$

a^b
- $a \in \mathbb{R}$ libovolné, $b \in \mathbb{N}$ $\quad (\underbrace{a \cdot a \cdots a}_{b})$
- $a \in \mathbb{R}, a \neq 0, \; b \in \mathbb{Z}, b < 0$

lze spočítat, že všechny definice spolu souhlasí:

dle nové def. „a^n" $= \exp(n \cdot \log a) = \exp(\log(a^n)) =$ „a^n" dle staré def.

$a > 0, n \in \mathbb{Z}$

- Pro obecnou mocninu platí známá početní pravidla, lze je odvodit z vlastností exp, log.

- $a \in \mathbb{R}, \quad f(x) = x^a, \; x \in (0, +\infty)$

$$\shortparallel$$

$\exp(a \cdot \log x)$ ... f je spojitá na $(0,+\infty)$ (skládání spojitých funkcí)

- $x^a \qquad x^{\frac{1}{3}} \diagup \exp(\frac{1}{3} \log x), \; x \in (0, +\infty)$

$\sqrt[3]{x}$, $x \in \mathbb{R}$

ÚMLUVA: x^a .... def. obor jen $(0,+\infty)$

$x^{\frac{1}{3}}$ má def. obor $(0,+\infty)$

$\sqrt[3]{x}$ .... inverzní k x^3 ... def. obor $\mathbb{R}$

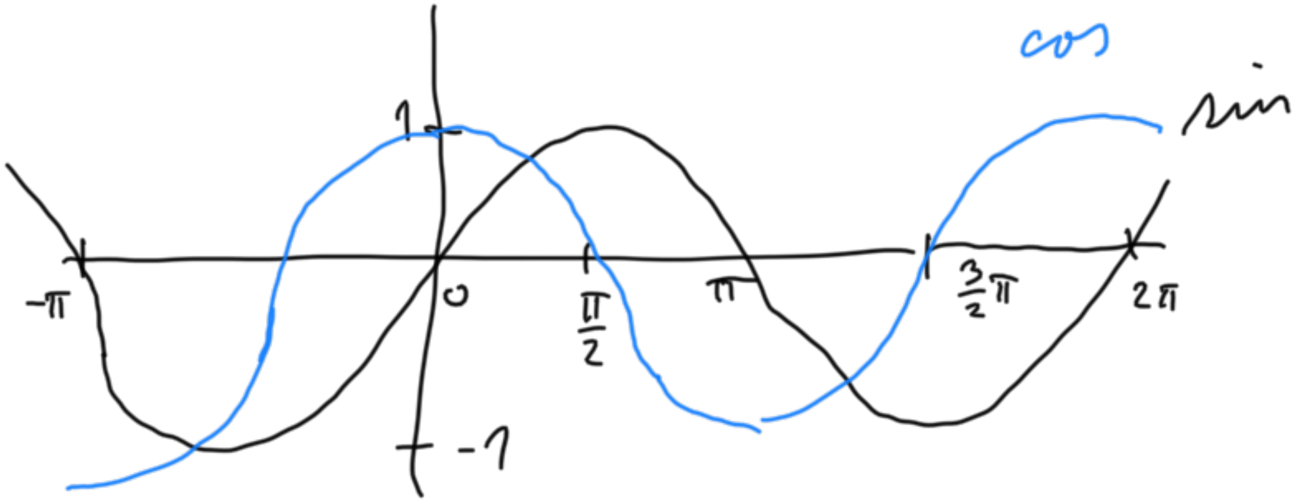
- $\log_a b$ ... „takové číslo, na které musíme umocnit a, abychom dostali b".

$$a^{\log_a b} \underset{\text{def.}}{=} \exp(\log_a b \cdot \log a) \underset{\text{def}}{=}$$

$$= \exp\left(\frac{\log b}{\log a} \cdot \log a\right) = \exp(\log b) = b$$

$$\log_e x = \frac{\log x}{\log e} = \frac{\log x}{1} = \log x$$

$(S4)$: $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$ .... známý součtový vzorec



∘ spojitost sinu:

a) v 0

$$\lim_{x\to 0}\sin x = \lim_{x\to 0}\left(x\cdot\frac{\sin x}{x}\right) \underset{\cancel{?}}{\overset{AL}{=}} \lim_{x\to 0} x \cdot \lim_{x\to 0}\frac{\sin x}{x} =$$

$$\overset{(S5)}{=} 0\cdot 1 = 0 \overset{(S3)}{=} \sin 0$$

b) $x_0 \in \mathbb{R}$ chci $\lim_{x\to x_0}\sin x = \sin x_0$:

$$\lim_{x\to x_0}(\sin x - \sin x_0) = \lim_{x\to x_0} 2\,\overbrace{\sin\frac{x-x_0}{2}}^{\to 0}\cdot\overbrace{\cos\frac{x+x_0}{2}}^{\in\langle -1,1\rangle} = 0$$

nulová × omezená

$$\lim_{x\to x_0}\underbrace{\sin\frac{x-x_0}{2}}_{} = f\circ g(x_0) = 0$$

$f \circ g(x)$ ... spoj. fce (složení spoj. fcí,

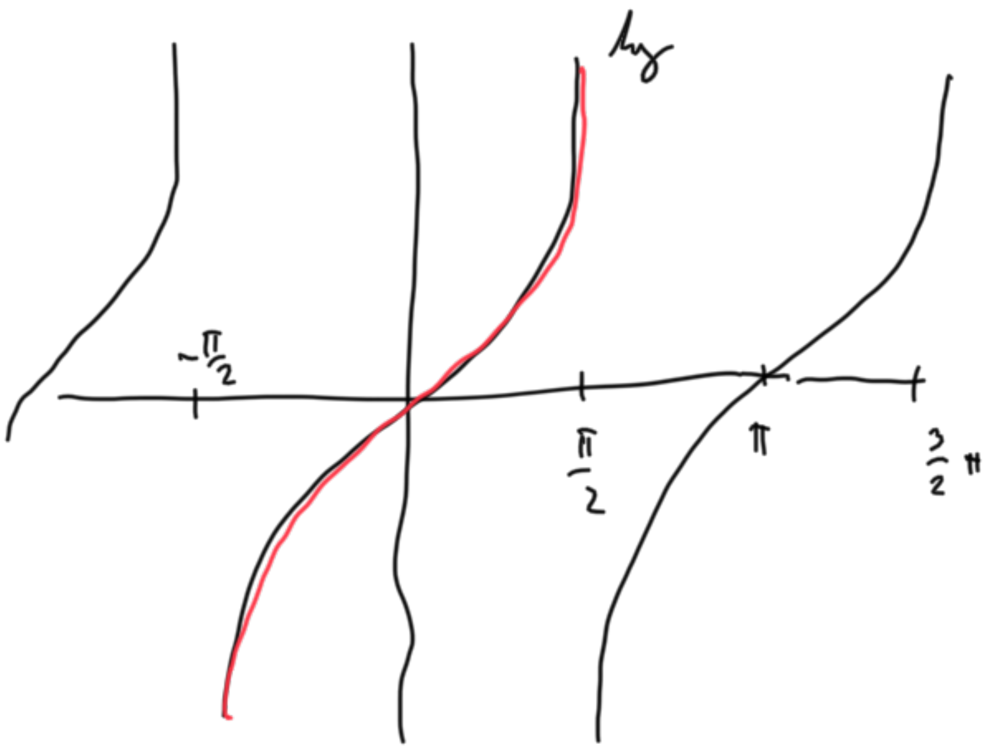
$f(y) = \sin y$ v x_0 g spoj. všude

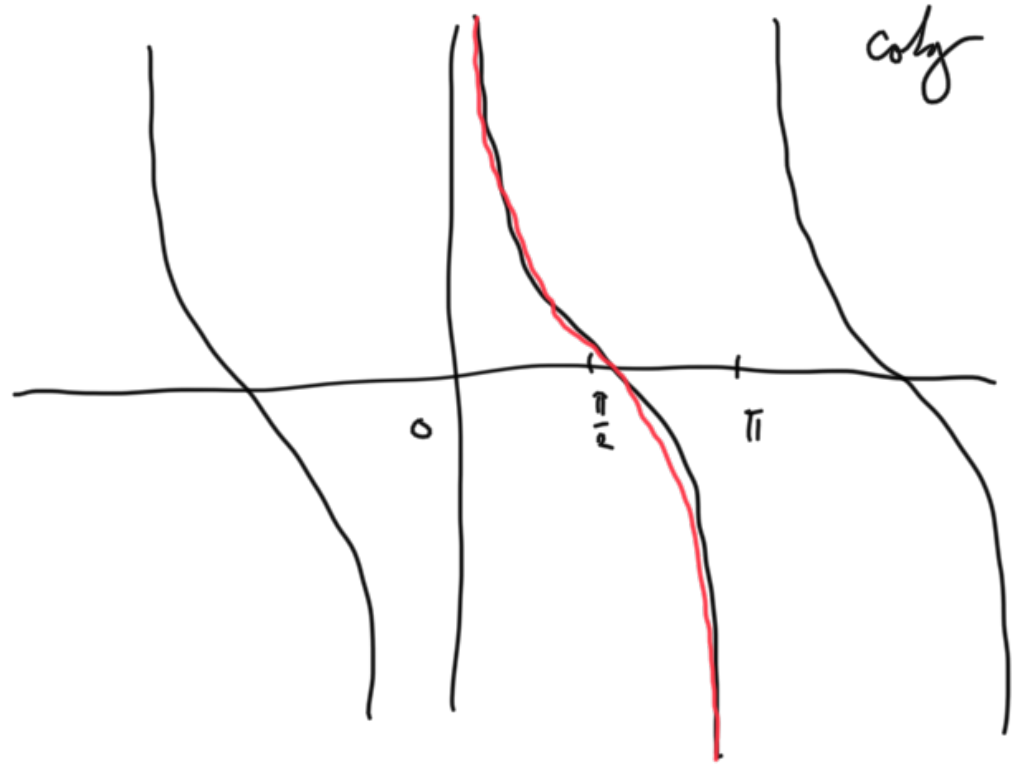
$g(x) = \frac{x-x_0}{2}$ f spoj. v 0 (ex a))

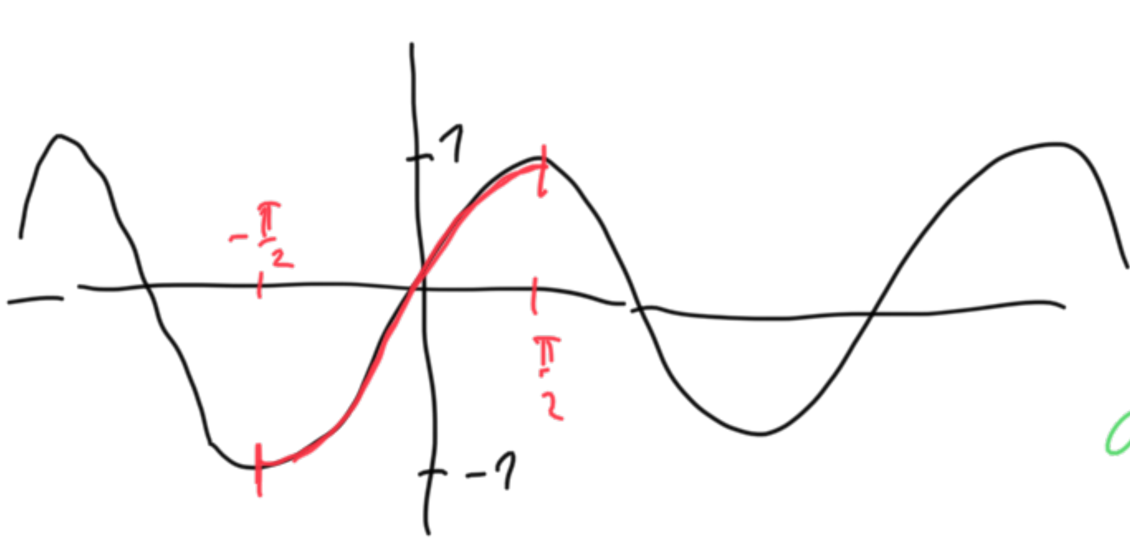
$g(x_0) = 0$

(u subst. „$\frac{x-x_0}{2} = y$")

π je iracionální číslo, $\pi \doteq 3{,}14159$





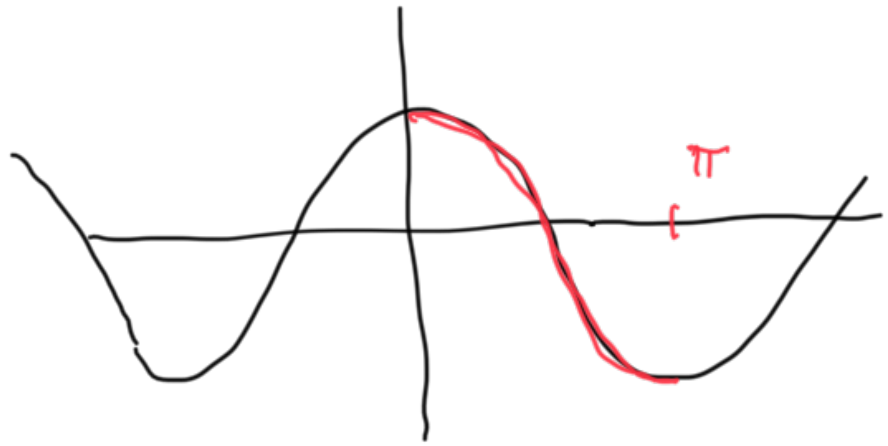


$\sin|_{\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle}$ je rostoucí na svém def. oboru $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$, zobrazuje jej na $\langle -1, 1 \rangle$

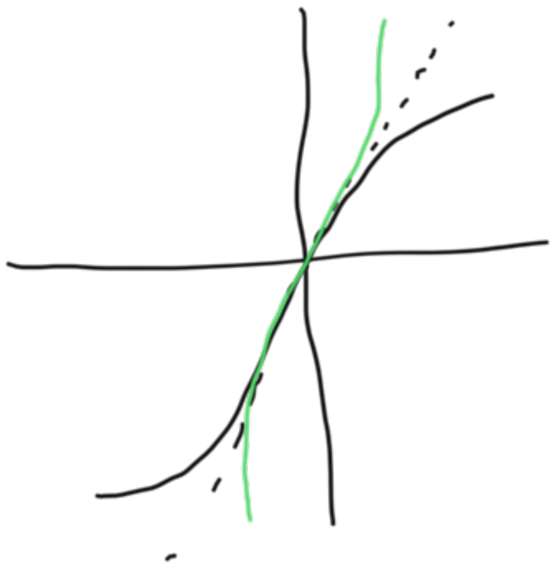
arcsin je fce inverzní k $\sin|_{\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle}$,

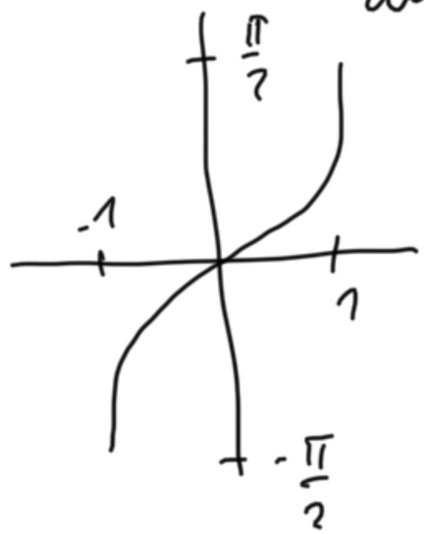
zobrazuje $\langle -1, 1 \rangle$ na $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

zobrazuje $\langle -1,1\rangle$ na $\langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$

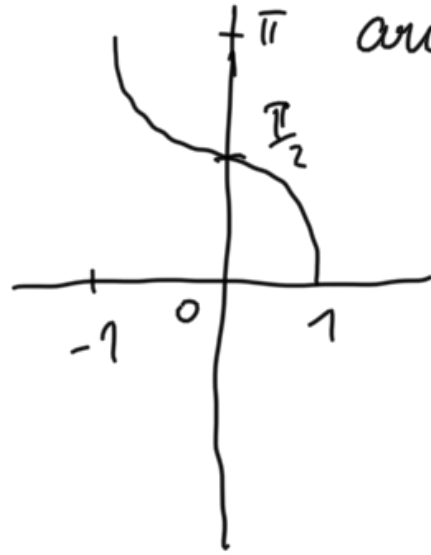


$\cos|_{\langle 0,\pi\rangle}$

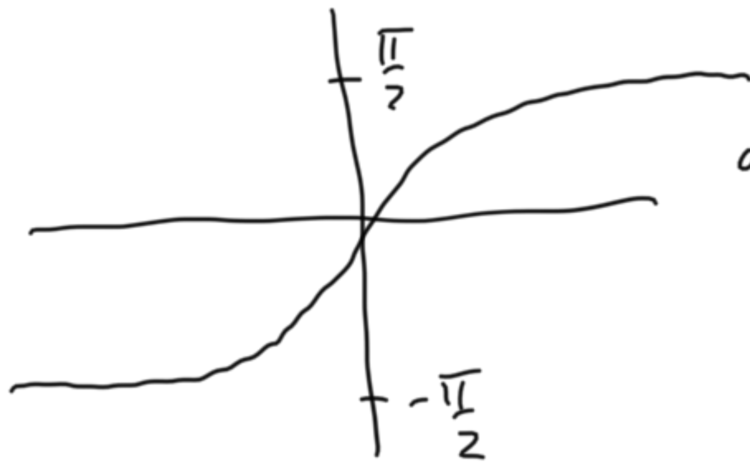




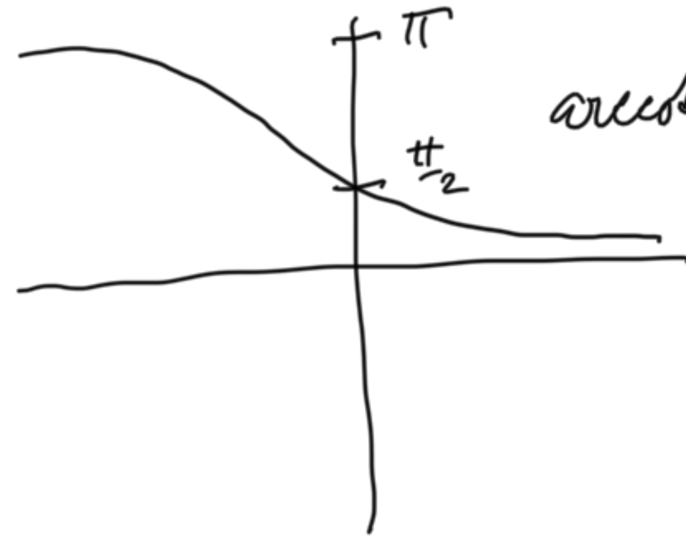
arcsin



arccos



arctg



arccotg

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{1}{f\circ g(x)} \overset{AL}{=} \frac{1}{1} = 1$$

$$f(y) = \frac{\sin y}{\quad} \qquad f\circ g(x) = \frac{\sin(\arcsin x)}{\quad} = \frac{x}{\quad}$$

$f(y) = \frac{\quad}{y}$ $\qquad$ arcsin x $\qquad$ arcsin x

$g(x) = \arcsin x$

$\lim\limits_{x \to 0} g(x) = \arcsin 0 = 0$

$\lim\limits_{y \to 0} f(y) = 1$

VOLBE — NENÍ splň. předp. (S); f není v 0 ani def.

(P): g je prostá (rostoucí)

$$\lim_{x \to 0} f \circ g(x) = 1$$

Varování: $\underbrace{\arcsin(\sin x)}_{\text{omezená na } \mathbb{R}} \neq \underbrace{x}_{\text{neomez. na } \mathbb{R}}$ $\qquad$ $\arcsin\left(\sin\big|_{\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle} x\right) = x$, tj.

$\hookrightarrow \forall x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

def. $\forall x \in \mathbb{R}$

Platí: $\arcsin(\sin x) = x$, $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$\sin(\arcsin x) = x$, $x \in \langle -1, 1 \rangle$