$\lim\limits_{x \to c} f(x)$, vím, že f je spojitá v c $\Rightarrow$ dosadím, tj. $\lim\limits_{x \to c} f(x) = f(c)$

$$\lim_{x \to 5} \underbrace{\frac{x^2+x}{x^3+1}}_{f(x)} = f(5) = \frac{5^2+5}{5^3+1} = \frac{30}{126} = \frac{5}{21}$$

f je spojitá $\forall x \in \mathbb{R}$ $x^3+1 \neq 0$

$x \neq -1$

tedy f je spojitá v 5

$$\lim_{x \to 1} \frac{x^2-1}{x^2-x} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x(x-1)} = \lim_{x \to 1} \frac{x+1}{x} = 2$$

(spoj. v 1)

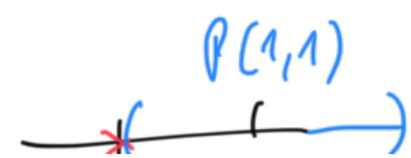
$$f(x) = \frac{(x-1)(x+1)}{x(x-1)} \qquad g(x) = \frac{x+1}{x}$$

$$D_f = \mathbb{R} \setminus \{0,1\} \qquad D_g = \mathbb{R} \setminus \{0\}$$

neplatí „$f = g$"

ale $f = g$ na nějakém prstencovém okolí 1

(např. $P(1,1)$)



0 1 2

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x} \underset{a^3-b^3}{=} \lim_{x\to 0} \frac{1+x-1^3}{(\sqrt[3]{1+x})^2+\sqrt[3]{1+x}\cdot 1+1^2}\cdot\frac{1}{x} =$$

$$= \lim_{x\to 0} \underbrace{\frac{1}{(\sqrt[3]{1+x})^2+\sqrt[3]{1+x}+1}}_{f(x)} \overset{AL}{=} \frac{1}{3}$$

$\sqrt[3]{1+x} \to 1$, $\sqrt[3]{1+x} \to 1$ spojitost $\sqrt[3]{\ }$

jiné zdůvodnění: f je spojitá v 0 ($\sqrt[3]{\ }$ spojitá, aritmetika spojitých, skládání spojitých)

můžu dosadit

$$\lim_{x\to -\infty} x\cdot\left(\sqrt{1+x^2}+x\right) \overset{AL}{=} (+\infty)\cdot 0$$

($\sqrt{1+x^2} \to +\infty$, $x \to -\infty$)

$\sqrt{1+x^2}+x \neq \sqrt{1+x^2}+\sqrt{x^2}$

∇ 0

$\sqrt{x^2}=|x|$

$\sqrt{1+x^2}+x = \sqrt{1+x^2}-|x| = \sqrt{1+x^2}-\sqrt{x^2} = \ldots$ $x<0$

$$= \frac{1+x^2-x^2}{\sqrt{1+x^2}+\sqrt{x^2}} = \frac{1}{\sqrt{1+x^2}+\sqrt{x^2}}$$

$$= \lim_{x\to-\infty} \frac{x}{\sqrt{1+x^2}+\sqrt{x^2}} = \lim_{x\to-\infty} \frac{x}{\sqrt{\frac{1}{x^2}+1}\cdot\sqrt{x^2}+\sqrt{x^2}} =$$

Dom. člen (x^2)

$\sqrt{x^2} \downarrow +\infty$, $\to +\infty$

$\sqrt{x^2}$: $\neq$ x !

$$= \lim_{x\to-\infty} \frac{x}{\left(\sqrt{\frac{1}{x^2}+1}+1\right)\cdot|x|} = \lim_{x\to-\infty} \frac{-1}{\sqrt{\frac{1}{x^2}+1}+1} =$$

$|x| = -x,\ x<0$

$\sqrt{\frac{1}{x^2}+1} \downarrow$ AL 1

$$\overset{AL}{=} \frac{-1}{1+1} = -\frac{1}{2}$$

VOLSF

$\sqrt{\ }$ spojitá

$\downarrow 1$

VOLSF

$f(y)=\sqrt{y}$

$g(x)=\frac{1}{x^2}+1$

(S) f je spojitá v 1

$\lim_{x\to-\infty} g(x) = 1$

$a_n \to a$

$\sqrt{a_n} \to \sqrt{a}$

~~$\lim g(x) = c$~~

~~$\lim \sqrt{g(x)} = \sqrt{c}$~~

$\lim \operatorname{sgn}(g(x)) = \operatorname{sgn} c$

VOLSF: „substituce pro limity"

$\sqrt{\underbrace{\frac{1}{x^2}+1}_{y}}$ $\quad \sqrt{y}$

$y = \frac{1}{x^2+1}$

$f(y)$ ... vnější fce

$f(y) \text{ „=" } f(g(x)) \to A$

$y = g(x)$

$\lim_{x \to c} g(x) \text{ „=" } \lim_{x \to c} y \text{"} = D$, tj.

„$x \to c \Rightarrow y \to D$"

„$f(y) \to A$"
$\quad y \to D$

$$\lim_{x \to +\infty} \frac{2x + \sqrt{x^2+1}}{x} = \lim_{x \to +\infty} \frac{2x + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x} =$$

$(+\infty) + (+\infty)$

$$= \lim_{x\to+\infty} \frac{2x + |x|\cdot\sqrt{1+\frac{1}{x^2}}}{x} \overset{x>0}{=} \lim_{x\to+\infty} \frac{2x + x\cdot\sqrt{1+\frac{1}{x^2}}}{x} =$$

$$= \lim_{x\to+\infty} 2+\sqrt{1+\frac{1}{x^2}} \overset{AL}{=} 2+1 = 3$$

$\sqrt{1+\frac{1}{x^2}} \to 1$

$\sqrt{\ }$ spojitá v 1

+ VOLSF

(vnější $y \mapsto \sqrt{y}$

vnitřní $x \mapsto 1+\frac{1}{x^2}$)

$$\lim_{x\to-\infty} \frac{2x+\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} \frac{2x+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x} =$$

$(-\infty) + (+\infty)$

$$= \lim_{x\to-\infty} \frac{2x+|x|\sqrt{1+\frac{1}{x^2}}}{x} \overset{x<0}{=} \lim_{x\to-\infty} \frac{2x-x\sqrt{1+\frac{1}{x^2}}}{x} =$$

$$= \lim_{x\to-\infty} 2-\sqrt{1+\frac{1}{x^2}} = 2-1 = 1$$

$$\lim \frac{2\cos^2 x + 3\cos x - 2}{\ldots} = \lim f(g(x)) = \lim f(y) = \lim \frac{2y^2+3y-2}{\ldots} =$$

($2\cos^2 x + 3\cos x - 2 \to 0$)

$\lim_{x\to\frac{\pi}{3}} \frac{\dots}{2\cos^2 x - 5\cos x + 2}$ ($2\cos^2 x - 5\cos x + 2 \to 0$) $= \lim_{x\to\frac{\pi}{3}} \dots = \lim_{y\to\frac{1}{2}} \frac{\dots}{2y^2-5y+2}$

↑ $y \to \frac{1}{2}$

VOLSF

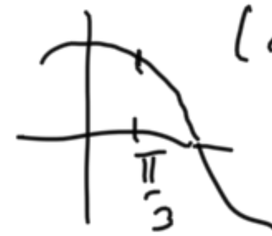
↑ $\lim_{x\to\frac{\pi}{3}} g(x) = \frac{1}{2}$ (cos je spojitý)

$\cos\frac{\pi}{3} = \frac{1}{2}$ „substituce" $\cos x = y$

$$\frac{2y^2+3y-2}{2y^2-5y+2} = f(y)$$

$g(x) = \cos x$

(P): cos je prostý na okolí $\frac{\pi}{3}$ (např. na $(0,\pi)$)

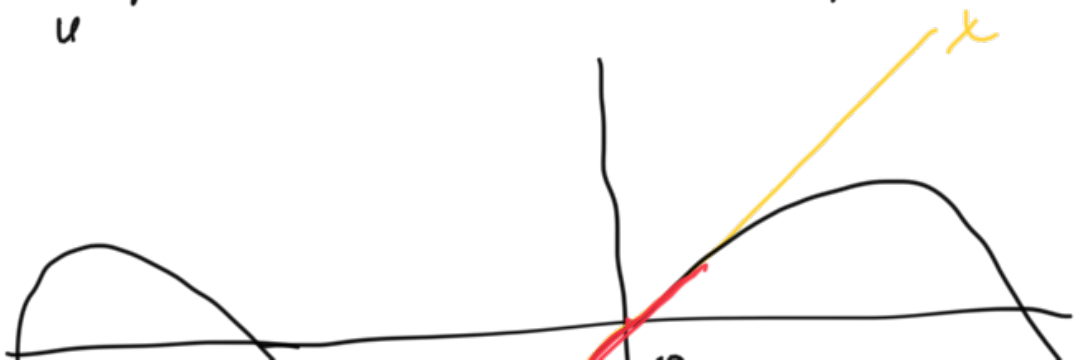


$$= \lim_{y\to\frac{1}{2}} \frac{y^2+\frac{3}{2}y-1}{y^2-\frac{5}{2}y+1} = \lim_{y\to\frac{1}{2}} \frac{(y-\frac{1}{2})(y+2)}{(y-\frac{1}{2})(y-2)} = \frac{2{,}5}{-1{,}5} = \frac{\frac{5}{2}}{-\frac{3}{2}} = -\frac{5}{3}$$

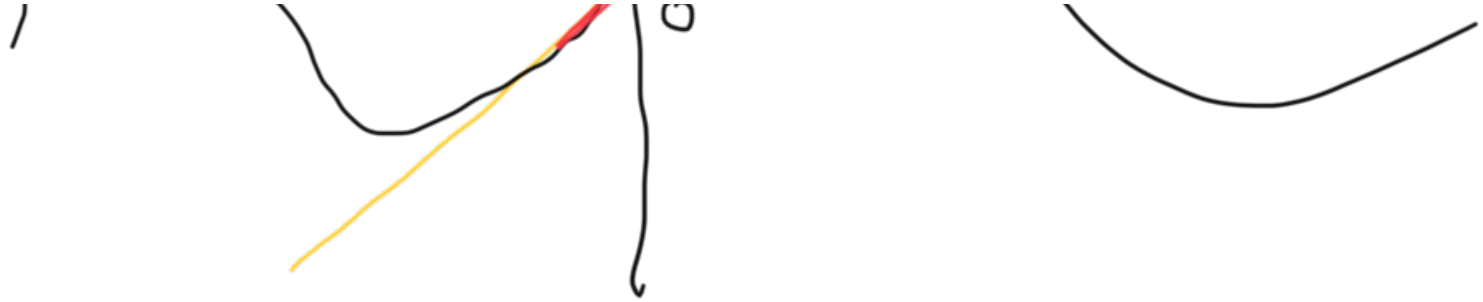
↑ spojitost

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ .... umožňuje „nahradit" při $\sin x$ „při x blízko 0, chyba bude malá"

„pro x blízko 0 je $\frac{\sin x}{x}$ skoro 1", tj. „přibližně $\sin x = x$"



„$\sin x$ vypadá blízko 0 skoro jako x"



$$\lim_{x\to 0} \frac{\sin(3x)}{x} \qquad „= \lim_{x\to 0} \frac{3x}{x} = 3"$$

$$= \lim_{x\to 0} \frac{\sin(3x)}{3x}\cdot 3 \overset{AZ}{=} 1\cdot 3 = 3$$

VOLSF

„subst. $y = 3x$"

$f(y) = \frac{\sin y}{y}$, $\lim_{y\to 0} f(y) = 1$ ... zákl. limita

$g(x) = 3x$, $\lim_{x\to 0} g(x) = 0$

$f(g(x)) = \frac{\sin 3x}{3x}$

(P) ... g je rostoucí

$g(x) = ax + b$, $a \neq 0$ .... prostá, je vždy splněna (P)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2}\cdot\frac{1}{1+\cos x} =$$

OK, protože $\cos x \neq -1$ na $(-\pi,\pi)$

$$= \lim_{x \to 0} \underbrace{\frac{\sin^2 x}{x^2}}_{\text{"}\left(\frac{\sin x}{x}\right)^2} \cdot \frac{1}{1+\cos x} \overset{AL}{\underset{\text{spoj. cos}}{=}} 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}$$

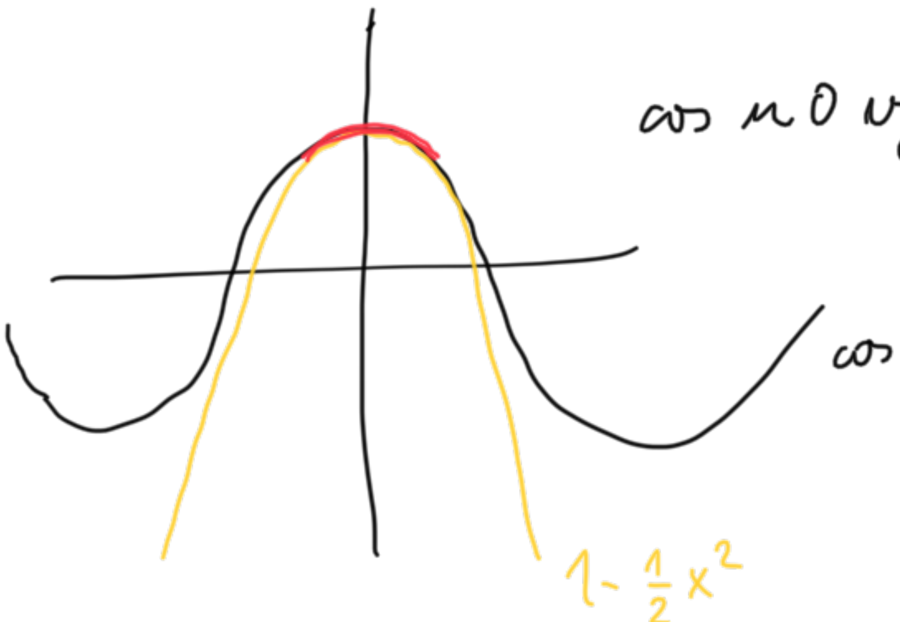
$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

„pro x blízko 0 je $\frac{1-\cos x}{x^2}$ skoro $\frac{1}{2}$", tj.

$1-\cos x$ je skoro $\frac{1}{2}x^2$

$\cos x$ je skoro $1-\frac{1}{2}x^2$

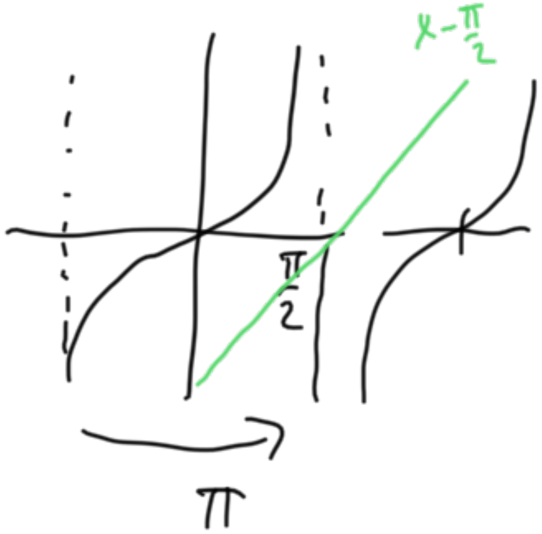
cos u 0 vypadá jako parabola



$$\lim_{x \to 0} \frac{\operatorname{tg} x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \frac{1}{\cos x} \overset{AL}{\underset{\text{spoj. cos}}{=}} 1 \cdot \frac{1}{1} = 1$$

$$\lim_{x\to\frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)\cdot \operatorname{tg} x = \lim_{x\to\frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)\cdot\frac{\sin x}{\cos x} \overset{AL}{=} \lim_{x\to\frac{\pi}{2}} \sin x \cdot \lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cos x} =$$

($\overset{AL}{=}$ crossed out; above $\lim_{x\to\frac{\pi}{2}} \sin x$: 1 || spoj. sin)



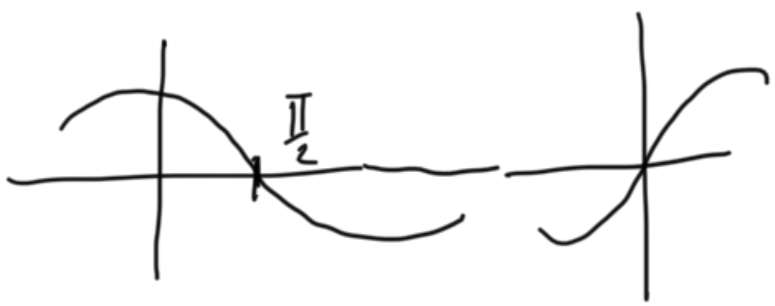
$$\left.\begin{array}{l}\lim_{x\to\frac{\pi}{2}-} \operatorname{tg} x = +\infty \\ \lim_{x\to\frac{\pi}{2}+} \operatorname{tg} x = -\infty\end{array}\right\} \lim_{x\to\frac{\pi}{2}} \operatorname{tg} x \text{ neexistuje}$$

$$\left(\lim_{x\to\frac{\pi}{2}} \frac{\operatorname{tg} x}{\operatorname{tg} x} = 1\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}-x\right)} = \lim_{x\to\frac{\pi}{2}} \frac{1}{-\boxed{\dfrac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}}} \overset{AL}{=} \frac{1}{-1} = -1$$

$\downarrow$ afinní subst. $y = \frac{\pi}{2} - x$

$\quad 1$



$k \in \mathbb{Z}$

$$\lim_{x\to 0} \frac{\sin(\sin(\sin x))}{\cos\left(\frac{\pi}{2}\cos x\right)} \cdot x^k = \lim_{x\to 0} \frac{\sin(\sin(\sin x))}{\sin(\sin x)} \cdot \frac{\sin(\sin x)}{\cos\left(\frac{\pi}{2}\cos x\right)} \cdot x^k =$$

„ošklivé" fce odstraníme ZVENKU

$\sin y \rightsquigarrow y$

↓ VOLSF

$1 \quad f(y) = \frac{\sin y}{y}$ | $\lim_{y\to 0} f(y) = 1$

$g(x) = \sin(\sin x)$ | $\lim_{x\to 0} g(x) = 0$

! f není def. v 0, tedy není splněno (S)

(P): g je rostoucí na $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(složení rost. fcí)

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-1, 1) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

---

jinak $\sin(\sin x) = 0 \iff \sin x = k\pi$

$\iff \sin x = 0$

$\iff x = k\pi$, k.

na $P(0, \pi)$ je $\sin(\sin x) \neq 0$

$$\overset{AL}{=} 1 \cdot \lim \frac{\sin(\sin x)}{\cos\left(\frac{\pi}{2}\cos x\right)} x^k = \lim \frac{\sin(\sin x)}{\ldots} \cdot \frac{\sin x}{\ldots} \cdot x^k$$

$$\lim_{x\to 0} \boxed{\sin x} \ \ldots \ \cos\left(\frac{\pi}{2}\cos x\right) =$$

$\downarrow$ VOLSF

1 vnější fce $\frac{\sin y}{y}$

vnitřní fce $\sin x$, rostoucí na $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tedy (P)

$$\overset{AL}{\underset{?}{=}} 1 \cdot \lim_{x\to 0} \frac{\sin x}{\cos\left(\frac{\pi}{2}\cos x\right)} x^k =$$

$$= \lim_{x\to 0} \boxed{\frac{\sin x}{x}} \cdot \frac{x^{k+1}}{\cos\left(\frac{\pi}{2}\cos x\right)} \overset{AL}{\underset{?}{=}} \lim_{x\to 0} \frac{x^{k+1}}{\cos\left(\frac{\pi}{2}\cos x\right)} = \lim_{x\to 0} \frac{x^{k+1}}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\cos x\right)} =$$

$\Downarrow$ 1 $\qquad \downarrow \frac{\pi}{2}$

$$= \lim_{x\to 0} \boxed{\frac{\frac{\pi}{2} - \frac{\pi}{2}\cos x}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\cos x\right)}} \cdot \frac{x^{k+1}}{\frac{\pi}{2} - \frac{\pi}{2}\cos x} \overset{AL}{\underset{?}{=}} 1 \cdot \lim_{x\to 0} \frac{x^{k+1}}{\frac{\pi}{2}(1-\cos x)} =$$

VOLSF $\downarrow$ 1

$f(y) = \frac{y}{\sin y}$ | $\lim_{y\to 0} f(y) \overset{AL}{=} \frac{1}{1} = 1$

$g(x) = \frac{\pi}{2} - \frac{\pi}{2}\cos x$ | $\lim g(x) = 0$

Podm. (P): $g(x)=0$

$$\frac{\pi}{2}=\frac{\pi}{2}\cos x$$

$$\cos x = 1$$

$$x = 2k\pi$$

tedy $\forall x \in P(0,2\pi): g(x)\neq 0$

$$= \lim_{x\to 0} \boxed{\frac{x^2}{1-\cos x}} \cdot \frac{x^{k+1}}{x^2}\cdot\frac{2}{\pi} \overset{AL}{=} \frac{2}{\pi}\cdot 2\cdot \lim_{x\to 0} x^{k-1} =$$

$\frac{x^2}{1-\cos x} \xrightarrow{AL} 2$

$$= \begin{cases} 0, & k>1 \quad (\text{AL, spojitost } x \mapsto x^{k-1}) \\ \frac{4}{\pi}, & k=1 \\ +\infty & k<1,\ k-1 \text{ sudé, tj. } k \text{ liché} \\ \text{neexistuje} & k<1,\ k \text{ sudé} \end{cases}$$

($\lim\limits_{x \to 0+} \dots = +\infty$, $\lim\limits_{x \to 0-} \dots = -\infty$ "$1/$kladná 0")

↳ celý výpočet by se provedl pro $\lim\limits_{x \to 0+}$, resp. $\lim\limits_{x \to 0-}$,
taktéž lze použít AL