I. Banach algebras

1. Basic properties

Definition 1. We say that $(A, +, -, 0, \cdot_{\mathrm{s}}, \cdot)$ is an *algebra* over $\mathbb{K}$ if $(A, +, -, 0, \cdot_{\mathrm{s}})$ is a vector space over $\mathbb{K}$, $(A, +, -, \cdot, 0)$ is a ring, and moreover $(\alpha \cdot_{\mathrm{s}} a) \cdot b = a \cdot (\alpha \cdot_{\mathrm{s}} b) = \alpha \cdot_{\mathrm{s}} (a \cdot b)$ for all $a, b \in A$ and $\alpha \in \mathbb{K}$. An algebra over $\mathbb{K}$ is called *commutative* if its ring multiplication $\cdot$ is commutative.

Proposition 2. *Let A be an algebra over $\mathbb{K}$. Put $A_{\mathrm{e}} = A \times \mathbb{K}$ and define vector operations on A_{e} in the usual way (i.e. componentwise) and further multiplication of the elements of A_{e} by the formula*

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad \text{for } a, b \in A,\ \alpha, \beta \in \mathbb{K}.$$

Then A_{e} is an algebra with the unit $(0, 1)$ and A can be identified with its subalgebra $A \times \{0\}$. If A is commutative, then so is A_{e}.

Let A, B be algebras over $\mathbb{K}$. (Algebra) homomorphism $\Phi\colon A \to B$ is a mapping which is a homomorphism between the respective vector spaces (i.e. it is linear) and also it is a homomorphism between the respective rings (i.e. it is multiplicative, or $\Phi(ab) = \Phi(a)\Phi(b)$).

Φ is called an (algebraic) isomorphism of algebras A and B if Φ is a bijection.

Fact 3. *Let A be an algebra, B an algebra with a unit e, and $\Phi\colon A \to B$ a homomorphism. Then $\widetilde{\Phi}\colon A_{\mathrm{e}} \to B$, $\widetilde{\Phi}(x, \lambda) = \Phi(x) + \lambda e$ is a homomorphism extending Φ.*

Proposition 4. *Let A be an algebra with a unit e and B a subalgebra of A not containing e. Then $C = B + \operatorname{span}\{e\}$ is a subalgebra of A and the mapping $\Phi\colon B_{\mathrm{e}} \to C$, $\Phi(x, \lambda) = x + \lambda e$ is an isomorphism.*

Definition 5. A pair $(A, \|\cdot\|)$ is called a *normed algebra* if A is an algebra, $(A, \|\cdot\|)$ is a normed linear space, and $\|ab\| \le \|a\|\|b\|$ for each $a, b \in A$. If the metric generated by $\|\cdot\|$ is complete, then $(A, \|\cdot\|)$ is called a *Banach algebra*.

Proposition 6. *Let $(A, \|\cdot\|)$ be a normed algebra. The multiplication of elements of A is Lipschitz on bounded sets (and in particular continuous) as a mapping from $A \times A$ to A.*

Corollary 7. *Let A be a normed algebra and B a subalgebra of A. Then $\overline{B}$ is also a subalgebra of A.*

Corollary 8. *Every normed algebra A has a completion, i.e. a Banach algebra such that A is its dense subalgebra. This completion is unique up to an isometry. If A has a unit e, then e is also a unit in the completion of A.*

Proposition 9. *Let $(A, \|\cdot\|)$ be a normed algebra. If we define a norm on A_{e} by the formula $\|(a, \alpha)\|_{A_{\mathrm{e}}} = \|a\| + |\alpha|$ (i.e. $A_{\mathrm{e}} = A \oplus_1 \mathbb{K}$), then A_{e} with this norm is a normed algebra. If $(A, \|\cdot\|)$ is a Banach algebra, then so is A_{e} with the norm above.*

Definition 10. Let A and B be normed algebras and $\Phi\colon A \to B$ an (algebra) homomorphism. We say that Φ is an isomorphism of normed algebras A and B (or just an *isomorphism*) if Φ is a homeomorphism of A onto B; we say that Φ is an isomorphism of A into B (or just an *isomorphism into*) if Φ is an isomorphism of A onto $\operatorname{Rng}\Phi$.

Theorem 11. *Let A be a normed algebra. For each $a \in A$ we define a left translation $L_a\colon A \to A$ by the formula $L_a(x) = ax$. Then $L_a \in \mathcal{L}(A)$ and the mapping $I\colon A \to \mathcal{L}(A)$, $I(a) = L_a$ is a continuous algebra homomorphism with $\|I\| \le 1$. If A has a unit e, then I is an isomorphism into and $I(e) = Id$. If $\|e\| = 1$ or $\|x^2\| = \|x\|^2$ for each $x \in A$ (e.g. if A is a subalgebra of $\ell_\infty(\Gamma)$), then I is an isometry into.*

Corollary 12. *Let $(A, \|\cdot\|)$ be a non-trivial normed algebra with a unit. Then there exists an equivalent norm $|||\cdot|||$ on A such that $(A, |||\cdot|||)$ is a normed algebra and $|||e||| = 1$.*

Recall that in a ring with a unit (or more generally in a monoid) the following holds: if x has a left and a right inverse, then these are equal (and it is then and inverse to x). In particular, inverses to invertible elements are uniquely determined. Further, the invertible elements form a group, i.e. if $x, y \in A$ are invertible, then also xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$. This group of invertible elements will be denoted by $A^\times$.

Fact 13. *Let A be an algebra with a unit and B its subalgebra containing e. Then $B^\times \subset A^\times \cap B$.*

Fact 14. *Let A, B be semigroups, $\Phi\colon A \to B$ a homomorphism onto, and let A be moreover a monoid with a unit e. Then B is a monoid with a unit $\Phi(e)$ and if $x \in A$ is invertible, then $\Phi(x)$ is invertible and $\Phi(x)^{-1} = \Phi(x^{-1})$. If moreover Φ is a bijection, then $\Phi\restriction_{A^\times}$ is an isomorphism of the groups $A^\times$ and $B^\times$.*

Lemma 15. *Let A be a normed algebra wit a unit and $x \in A$. If the series $\sum_{n=0}^\infty x^n$ converges, then $\sum_{n=0}^\infty x^n = (e - x)^{-1}$.*

Lemma 16. *Let A be a Banach algebra with a unit.*

(a) If $x \in U_A$, then the series $\sum_{n=0}^\infty x^n$ converges absolutely and so $\sum_{n=0}^\infty x^n = (e - x)^{-1}$.

(b) *Let $x \in A^\times$ and let $h \in A$ be such that $\|h\| < \frac{1}{\|x^{-1}\|}$. Then $x+h \in A^\times$. If moreover $\|h\| \leq \frac{1}{2\|x^{-1}\|}$, then $\left\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}\right\|$ $2\|x^{-1}\|^3\|h\|^2$.*

Definition 17. Let G be a group and τ a topology on G. We say that (G, τ) is a *topological group* if the group operations (i.e. multiplication $\cdot\colon G \times G \to G$ and inversion ${}^{-1}\colon G \to G$) are continuous.

Theorem 18. *Let A be a Banach algebra with a unit. Then $A^\times$ is an open subset of A and it is a topological group.*

Proposition 19. *Let A be a Banach algebra with a unit and B its closed subalgebra containing e. Then $(\partial_B B^\times) \cap A^\times = \emptyset$ and*

$$B^\times = \bigcup \{C \subset B;\ C \text{ is a component of } A^\times \cap B \text{ intersecting } B^\times\}.$$

2. Spectral theory

Definition 20. Let A be an algebra with a unit. For $x \in A$ we define the *resolvent set* of x as

$$\rho(x) = \{\lambda \in \mathbb{K};\ \lambda e - x \in A^\times\}$$

and the *spectrum* of x as

$$\sigma(x) = \mathbb{K} \setminus \rho(x).$$

If A does not have a unit, then for $x \in A$ we define the above notions with respect to the algebra A_e.

Definition 21. An element x of a groupoid is called *idempotent* if $x^2 = x$.

Proposition 22. *Let A, B be algebras and $\Phi\colon A \to B$ an algebraic isomorphism. Then $\sigma(\Phi(x)) = \sigma(x)$ for every $x \in A$.*

Lemma 23. *Let M be a monoid and $x, y \in M$. If at least two of the elements x, y, xy, and yx are invertible, then all four are invertible.*

Proposition 24. *Let A be an algebra over $\mathbb{K}$.*

(a) *If A is non-trivial, then $\sigma(0) = \{0\}$.*

(b) *If A has a unit, then $\sigma(\alpha e + \beta x) = \alpha + \beta\sigma(x)$ for every $x \in A$ and $\alpha, \beta \in \mathbb{K}$.*

(c) *If $x \in A$, $n \in \mathbb{N}$, and $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$.*

(d) *If $x \in A^\times$, then $\lambda \in \sigma(x)$ if and only if $\frac{1}{\lambda} \in \sigma(x^{-1})$.*

(e) *If $x, y \in A$, then the sets $\sigma(xy)$ and $\sigma(yx)$ differ at most by the element 0. If moreover $x \in A^\times$, then $\sigma(xy) = \sigma(yx)$.*

(f) *If $z \in A^\times$, then $\sigma(x) = \sigma(zxz^{-1})$ for every $x \in A$.*

Proposition 25. *Let X, Y be normed linear spaces, $T \in \mathcal{L}(X)$, and let $S\colon X \to Y$ be a linear isomorphism. Then the operator $S \circ T \circ S^{-1} \in \mathcal{L}(Y)$ has the following property: $\sigma(S \circ T \circ S^{-1}) = \sigma(T)$ a $\sigma_\mathrm{p}(S \circ T \circ S^{-1}) = \sigma_\mathrm{p}(T)$.*

Fact 26. *Let A be an algebra and B an ideal in A. Then B is also an ideal in A_e.*

Proposition 27. *Let A be an algebra.*

(a) *$0 \in \sigma_{A_\mathrm{e}}(x)$ for every $x \in A$. So, if A does not have a unit, then $0 \in \sigma(x)$ for every $x \in A$.*

(b) *If A has a unit, then $\sigma_{A_\mathrm{e}}(x) = \sigma_A(x) \cup \{0\}$ for every $x \in A$.*

(c) *Suppose that A has a unit e, B is a subalgebra of A not containing e, and $C = B + \operatorname{span}\{e\}$. Then $\sigma_C(x) = \sigma_{B_\mathrm{e}}(x)$ for every $x \in B$.*

(d) *Let B be a subalgebra of A and $x \in B$. If B has a unit which is not a unit in A, then $\sigma_A(x) \subset \sigma_B(x) \cup \{0\}$, in the other cases $\sigma_A(x) \subset \sigma_B(x)$.*

(e) *If B is a proper ideal in A, then $\sigma_{B_\mathrm{e}}(x) = \sigma_A(x)$ for every $x \in B$.*

Proposition 28. *Let A, B be algebras, $\Phi\colon A \to B$ a homomorphism, and $x \in A$. If A has a unit e and $\Phi(e)$ is not a unit in B, then $\sigma_B(\Phi(x)) \subset \sigma_A(x) \cup \{0\}$, in the other cases $\sigma_B(\Phi(x)) \subset \sigma_A(x)$.*

Definition 29. Let A be an algebra. For $x \in A$ we define the *spectral radius* of x as

$$r(x) = \sup\{|\lambda| \in [0, +\infty);\ \lambda \in \sigma(x)\}.$$

Theorem 30. *Let A be a Banach algebra and $x \in A$. Then $\rho(x)$ is open, $\sigma(x)$ is compact, and*

$$r(x) \le \inf_{n\in\mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Lemma 31. *Let $\{a_n\}$ be a sequence of real numbers.*

(a) If $a_{m+n} \le a_m + a_n$ for all $m, n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \frac{a_n}{n} = \inf\limits_{n\in\mathbb{N}} \frac{a_n}{n} < +\infty$.

(b) If $\{a_n\}$ is non-negative and $a_{m+n} \le a_m a_n$ for all $m, n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \inf\limits_{n\in\mathbb{N}} \sqrt[n]{a_n} \in \mathbb{R}$.

Theorem 32. *Let A be a Banach algebra with a unit, B its closed subalgebra containing e, and $x \in B$. Then the following hold:*

(a) $\partial\rho_B(x) \subset \partial\rho_A(x)$ and

$$\rho_B(x) = \bigcup\{C \subset \mathbb{K};\ C \text{ is a component of } \rho_A(x) \text{ intersecting } \rho_B(x)\}.$$

(b) If C is a component of $\rho_A(x)$, then either $C \subset \sigma_B(x)$, or $C \cap \sigma_B(x) = \emptyset$. Further, $\partial\sigma_B(x) \subset \partial\sigma_A(x)$.

(c) If $\rho_A(x)$ is connected, then $\sigma_B(x) = \sigma_A(x)$.

(d) If $\sigma_B(x)$ has an empty interior, then $\sigma_B(x) = \sigma_A(x)$.

Definition 33. Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, and $a \in \Omega$. If $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} \in Y$ exists, then this limit is called the *derivative* of the mapping f at a and it is denoted by $f'(a)$.

Fact 34. *Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, and $a \in \Omega$. If $f'(a)$ exists, then $(\phi \circ f)'(a) = \phi(f'(a))$ for every $\phi \in Y^*$.*

Fact 35. *Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, and $a \in \Omega$. If $f'(a)$ exists, then f is continuous at a.*

Definition 36. Let A be an algebra over $\mathbb{K}$ with a unit. On $\rho(x)$ we define the *resolvent* (or the *resolvent mapping*) of the element x by the formula

$$R_x(\lambda) = (\lambda e - x)^{-1}, \quad \lambda \in \rho(x).$$

If A does not have a unit, then we define the resolvent with respect to the algebra A_e.

Proposition 37. *Let A be a Banach algebra and $x \in A$. Then the mapping $\lambda \mapsto R_x(\lambda)$ has a derivative at every point of the set $\rho(x)$.*

Definition 38. Let Y be a complex normed linear space, $\Omega \subset \mathbb{C}$ an open set, and $f : \Omega \to Y$. We say that f is *holomorphic* on Ω, if $f'(z)$ exists for every $z \in \Omega$.

Theorem 39 (Liouville's theorem)**.** *Let Y be a complex normed linear space and let $f : \mathbb{C} \to Y$ be holomorphic on $\mathbb{C}$. If f is bounded, then it is constant.*

Theorem 40. *Let A be a complex Banach algebra and $x \in A$.*

(a) The resolvent mapping R_x is holomorphic on $\rho(x)$.

(b) If A is non-trivial, then $\sigma(x) \neq \emptyset$.

(c) $r(x) = \inf\limits_{n\in\mathbb{N}} \sqrt[n]{\|x^n\|} = \lim\limits_{n\to\infty} \sqrt[n]{\|x^n\|}$ (the Beurling-Gelfand formula).

Corollary 41. *If A is a complex Banach algebra, $x \in A$, and $\lambda \in \mathbb{C}$, $|\lambda| > r(x)$, then the sum $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ converges absolutely. So if A has a unit, then $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.*

Theorem 42 (S. Mazur (1938), I. M. Gelfand (1941))**.** *Let A be a non-trivial complex Banach algebra with a unit. If $A^\times = A\setminus\{0\}$, then A is isomorphic to $\mathbb{C}$. If moreover $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.*

3. Holomorphic calculus

Let A be a Banach algebra over $\mathbb{K}$ with a unit and $x \in A$. Further let $\mathcal{F}$ be some algebra of functions defined on a subset of $\mathbb{K}$ that contains polynomials. A functional calculus for x will be some homomorphism $\Phi\colon \mathcal{F} \to A$ such that $\Phi(1) = e$, $\Phi(Id) = x$, and which is moreover continuous, resp. sequentially continuous, in some convenient topologies on $\mathcal{F}$ and A.

Theorem 43. *Let A be a complex algebra with a unit and $x \in A$. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open neighbourhoods of $\sigma(x)$ and let $\Phi_i\colon H(\Omega_i) \to A$ be an algebra homomorphism such that $\Phi_i(1) = e$, $\Phi_i(Id) = x$, and Φ_i is sequentially continuous from the topology of locally uniform convergence on $H(\Omega_i)$ to some Hausdorff topology τ on A, $i = 1, 2$. If $f_i \in H(\Omega_i)$, $i = 1, 2$ are such that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$, then $\Phi_1(f_1) = \Phi_2(f_2)$.*

Let X be a complex Banach space, $\gamma\colon [a, b] \to \mathbb{C}$ a path, and $f\colon \langle\gamma\rangle \to X$ a continuous mapping. The integral of f along γ is defined by

$$\int_\gamma f = \int_{[a,b]} \gamma'(t) f(\gamma(t))\, \mathrm{d}\lambda(t).$$

The integral along a chain $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ in $\mathbb{C}$ of a continuous mapping $f\colon \langle\Gamma\rangle \to X$ is defined by

$$\int_\Gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

Lemma 44. *Let Γ be a chain in $\mathbb{C}$, X a complex Banach space, $f\colon \langle\Gamma\rangle \to X$ continuous, and $\phi \in X^*$. Then $\phi\big(\int_\Gamma f\big) = \int_\Gamma \phi \circ f$.*

If $\Omega \subset \mathbb{C}$ is open and $K \subset \Omega$ compact, then we say that a cycle Γ surrounds K in Ω if $\langle\Gamma\rangle \subset \Omega \setminus K$, $\operatorname{ind}_\Gamma z = 1$ for $z \in K$, and $\operatorname{ind}_\Gamma z = 0$ for $z \in \mathbb{C} \setminus \Omega$.

Theorem 45. *Let $\Omega \subset \mathbb{C}$ be open, X a complex Banach space, and let $f\colon \Omega \to X$ be holomorphic. If Γ_1, Γ_2 are two cycles in Ω such that $\operatorname{ind}_{\Gamma_1}(z) = \operatorname{ind}_{\Gamma_2}(z)$ for every $z \in \mathbb{C} \setminus \Omega$, then $\int_{\Gamma_1} f = \int_{\Gamma_2} f$.*

Definition 46. Let A be a complex Banach algebra with a unit and $x \in A$. If $f \in H(\Omega)$, where $\Omega \subset \mathbb{C}$ is an open neighbourhood of $\sigma(x)$, then we define

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f R_x = \frac{1}{2\pi i}\int_\Gamma f(\alpha)(\alpha e - x)^{-1}\, \mathrm{d}\alpha,$$

where Γ is any cycle surrounding $\sigma(x)$ in Ω.

Lemma 47. *Let (Ω, μ) be a space with a complete measure, A a Banach algebra and $f \in L_1(\mu, A)$. Then*

$$x\Big(\int_E f d\mu\Big) = \int_E x f(t)\, \mathrm{d}\mu(t) \quad \text{and} \quad \Big(\int_E f\, \mathrm{d}\mu\Big)x = \int_E f(t)x\, \mathrm{d}\mu(t)$$

for every $x \in A$ and every measurable $E \subset \Omega$.

Fact 48. *Let G be a group. If $u, v \in G$ commute, then also u, v, u^{-1}, v^{-1} commute.*

Lemma 49. *Let A be an algebra with a unit, $x \in A$, and $\mu, \nu \in \rho(x)$.*

(a) $R_x(\mu)R_x(\nu) = R_x(\nu)R_x(\mu)$.

(b) $R_x(\mu) - R_x(\nu) = (\nu - \mu)R_x(\mu)R_x(\nu)$ (resolvent identity).

Theorem 50 (holomorphic calculus)**.** *Let A be a complex Banach algebra with a unit, $x \in A$, $\Omega \subset \mathbb{C}$ an open neighbourhood of $\sigma(x)$, and $f \in H(\Omega)$. The mapping $\Phi\colon H(\Omega) \to A$, where $\Phi(g) = g(x)$ from Definition 46, has the following properties:*

(a) Consider $H(\Omega)$ with the topology of locally uniform convergence. Then Φ is a continuous algebra homomorphism for which $\Phi(1) = e$ and $\Phi(Id) = x$.

(b) $f(x) \in A^\times$ if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$. In this case $f(x)^{-1} = \frac{1}{f}(x)$.

(c) $\sigma(f(x)) = f(\sigma(x))$ (spectral mapping theorem).

(d) If $g \in H(\Omega_1)$, where Ω_1 is an open neighbourhood of $f(\sigma(x))$, then $(g \circ f)(x) = g(f(x))$.

(e) If $y \in A$ commutes with x, then y commutes also with $f(x)$.

(f) If B is a complex Banach algebra and $\Theta\colon A \to B$ a continuous homomorphism such that $\Theta(e)$ is a unit in B, then $f(\Theta(x)) = \Theta(f(x))$. In particular, if $z \in A^\times$, then $f(zxz^{-1}) = zf(x)z^{-1}$.

4. Multiplicative linear functionals

Definition 51. Let A be an algebra over $\mathbb{K}$. A homomorphism $\varphi\colon A \to \mathbb{K}$ is called a *multiplicative linear functional* (i.e. φ is linear and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$). The set of all non-zero multiplicative linear functionals on A is denoted by $\Delta(A)$.

Proposition 52. *Let A be an algebra over $\mathbb{K}$. Then $\Delta(A)$ is a linearly independent set.*

Proposition 53. *Let A be an algebra. Every multiplicative linear functional φ on A has a unique extension $\widetilde{\varphi} \in \Delta(A_e)$ given by $\widetilde{\varphi}(x, \lambda) = \varphi(x) + \lambda$ and $\Delta(A_e) = \{\widetilde{\varphi};\ \varphi \in \Delta(A) \cup \{0\}\}$.*

Proposition 54. *Let A be an algebra and $\varphi \in \Delta(A)$. Then $\varphi(x) \in \sigma(x)$ for every $x \in A$ and so $|\varphi(x)| \leq r(x)$.*

Corollary 55. *Let A be a Banach algebra. Then $\Delta(A) \subset B_{A^*}$ (in particular, every multiplicative linear functional on A is automatically continuous). If A has a unit, then $\|\varphi\| \geq \frac{1}{\|e\|}$ for every $\varphi \in \Delta(A)$. In particular, if $\|e\| = 1$, then $\Delta(A) \subset S_{A^*}$.*

Definition 56. Let A be an algebra. A *maximal ideal* in A is a proper ideal in A that is maximal with respect to the ordering of all proper ideals in A by inclusion.

Proposition 57. *Let A be an algebra with a unit. Then every proper ideal in A is contained in some maximal ideal in A.*

Proposition 58. *Let A be a Banach algebra with a unit. If I is a proper ideal in A, then also $\overline{I}$ is a proper ideal in A. So every maximal ideal in A is closed.*

Lemma 59. *Let A be a commutative algebra with a unit and suppose that $x \in A$ is not invertible. Then the principal ideal xA is proper.*

Theorem 60. *Let A be a complex commutative Banach algebra with a unit and let I be a proper ideal in A. Then there exists $\varphi \in \Delta(A)$ such that $\varphi\restriction_I = 0$.*

Corollary 61. *If A is a non-trivial complex commutative Banach algebra with a unit, then $\Delta(A) \neq \emptyset$.*

Corollary 62. *Let A be a complex commutative Banach algebra with a unit. Then the mapping $\Phi\colon \varphi \mapsto \operatorname{Ker}\varphi$ is a bijection between $\Delta(A)$ and the set of all maximal ideals in A.*

Theorem 63. *Let A be a Banach algebra and $M = \Delta(A) \cup \{0\} \subset (B_{A^*}, w^*)$ is the set of all linear multiplicative functionals on A. Then M is compact, $\Delta(A)$ is locally compact, and if A has a unit, then $\Delta(A)$ is compact. If $\Delta(A)$ is not compact, then M is the Alexandrov compactification of $\Delta(A)$.*

The mapping $\Phi\colon M \to \Delta(A_e)$, where $\Phi(\varphi) = \widetilde{\varphi}$ is the unique extension of φ to the element of $\Delta(A_e)$, is a homeomorphism.

Let X, Y be vector spaces and $T\colon X \to Y$ be a linear mapping. Then we define the algebraically dual mapping $T^\#\colon Y^\# \to X^\#$ by the formula $T^\# f(x) = f(Tx)$ for $f \in Y^\#$ and $x \in X$.

Lemma 64. *Let X, Y be vector spaces and $T\colon X \to Y$ a linear bijection. Then $T^\#$ is a bijection and $(T^\#)^{-1} = (T^{-1})^\#$.*

Proposition 65. *Let A, B be Banach algebras and $\Phi\colon A \to B$ an algebraic isomorphism. Then the mapping $\Psi = \Phi^\#\restriction_{\Delta(B)}$ is a homeomorphism of $\Delta(B)$ onto $\Delta(A)$.*

Proposition 66. *Let S, T be topological spaces and let $h\colon S \to T$ be continuous and onto. Then $\Phi\colon C_b(T) \to C_b(S)$, $\Phi(f) = f \circ h$ is an isometric isomorphism of the Banach algebra $C_b(T)$ into the Banach algebra $C_b(S)$. If S and T are locally compact Hausdorff spaces and h is a homeomorphism, then $\Phi\restriction_{C_0(T)}$ is an isometric isomorphism of Banach algebras $C_0(T)$ and $C_0(S)$.*

Theorem 67. *Let K, L be locally compact Hausdorff topological spaces. Then the following statements are equivalent:*

(i) The Banach algebras $C_0(K)$ and $C_0(L)$ are isometrically isomorphic.

(ii) The algebras $C_0(K)$ and $C_0(L)$ are algebraically isomorphic.

(iii) The spaces K and L are homeomorphic.

Definition 68. A *commutative* algebra A is called *semi-simple* if $\Delta(A)$ separates the points of A, i.e. if $\bigcap\{\operatorname{Ker}\varphi;\ \varphi \in \Delta(A)\} = \{0\}$.

Theorem 69. *Let A, B be Banach algebras and suppose B is commutative and semi-simple. Then every homomorphism from A to B is automatically continuous. Also every conjugate-linear multiplicative mapping from A to B is automatically continuous.*

Corollary 70. *Let A be a commutative semi-simple algebra. Then all norms on A in which A is a Banach algebra are equivalent.*

5. Gelfand transform

Definition 71. Let A be a Banach algebra over $\mathbb{K}$. For $x \in A$ we define $\widehat{x}\colon \Delta(A) \to \mathbb{K}$ by the formula $\widehat{x}(\varphi) = \varphi(x)$, i.e. $\widehat{x} = \varepsilon_x \restriction_{\Delta(A)}$. The function $\widehat{x}$ is called the *Gelfand transform* of the element x.

Theorem 72. *Let A be a complex commutative Banach algebra and $x \in A$. If A has a unit, then* $\operatorname{Rng}\widehat{x} = \sigma(x)$. *If A does not have a unit, then $\sigma(x) \setminus \{0\} \subset \operatorname{Rng}\widehat{x} \subset \sigma(x)$.*

Corollary 73. *Let A be a complex commutative Banach algebra and $x \in A$. Then $\|\widehat{x}\|_{C_0(\Delta(A))} = r(x)$.*

Definition 74. Let A be a Banach algebra. The mapping $\Gamma\colon A \to C_0(\Delta(A))$, $\Gamma(x) = \widehat{x}$ is called the *Gelfand transform* of the algebra A.

Proposition 75. *Let A be a Banach algebra and let Γ be its Gelfand transform. Then the following hold:*

(a) Γ is a continuous homomorphism and $\|\Gamma\| \le 1$.

(b) The subalgebra $\Gamma(A) \subset C_0(\Delta(A))$ separates the points of $\Delta(A)$.

(c) Γ is one-to-one if and only if $\Delta(A)$ separates the points of A, i.e. if and only if A is commutative and semi-simple.

Theorem 76. *Let A be a complex commutative Banach algebra and let Γ be its Gelfand transform. Then the following hold:*

(a) Γ is an isomorphism into if and only if there exists $K > 0$ such that $\|x^2\| \ge K\|x\|^2$ for every $x \in A$.

(b) Γ is an isometry into if and only if $\|x^2\| = \|x\|^2$ for every $x \in A$.

Definition 77. Let A be a groupoid and $M \subset A$. Then the set $M^{\mathrm{c}} = \{a \in A;\ ax = xa \text{ for every } x \in M\}$, i.e. the set of all elements of A commuting with every element of M, is called the *commutant* of the set M.

Proposition 78. *Let A be a groupoid and $M \subset A$. Then the following hold:*

(a) $M \subset (M^{\mathrm{c}})^{\mathrm{c}}$.

(b) The set $M \cap M^{\mathrm{c}}$ commutes.

(c) If M commutes, then also $(M^{\mathrm{c}})^{\mathrm{c}}$ commutes.

Proposition 79. *Let A be an algebra and $M \subset A$. Then the following hold:*

(a) M^{c} is a subalgebra of A.

(b) If A has a unit, then $e \in M^{\mathrm{c}}$.

(c) If A is normed, then M^{c} is closed.

Proposition 80. *Let A be an algebra with a unit e and suppose that $M \subset A$ commutes. Then $B = (M^{\mathrm{c}})^{\mathrm{c}}$ is a commutative algebra with a unit e, $M \subset B$, and $B^\times = A^\times \cap B$. So $\sigma_A(x) = \sigma_B(x)$ for every $x \in B$.*

Theorem 81. *Let A be a complex Banach algebra and suppose that $x, y \in A$ commute. Then the following hold:*

(a) $\sigma(x+y) \subset \sigma(x) + \sigma(y)$ and $\sigma(xy) \subset \sigma(x)\sigma(y)$.

(b) $r(x+y) \le r(x) + r(y)$ and $r(xy) \le r(x)r(y)$.

6. B*-algebras

Theorem 82. *Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then there exists a unique operator $T^\star \in \mathcal{L}(H_2, H_1)$ such that*

$$\langle Tx, y\rangle_{H_2} = \langle x, T^\star y\rangle_{H_1}$$

for every $y \in H_2$ and $x \in H_1$. Further, $T^\star = I_1^{-1} \circ T^* \circ I_2$, where $I_j\colon H_j \to H_j^*$, $j = 1,2$ are the corresponding conjugate-linear isometries from the Löwig-Fréchet-Riesz theorem.

Definition 83. The operator $T^\star$ from the preceding theorem is called the *hilbertian adjoint operator* to T.

Theorem 84. *Let H_1, H_2, H_3 be Hilbert spaces.*

(a) If $T \in \mathcal{L}(H_1, H_2)$, then $T^{\star\star} = (T^\star)^\star = T$.

(b) The mapping $T \mapsto T^\star$ is a conjugate-linear isometry of $\mathcal{L}(H_1, H_2)$ onto $\mathcal{L}(H_2, H_1)$.

(c) Let $T \in \mathcal{L}(H_1, H_2)$ and $S \in \mathcal{L}(H_2, H_3)$. Then $(S \circ T)^\star = T^\star \circ S^\star$. Also, $(Id_{H_1})^\star = Id_{H_1}$.

(d) Let $T \in \mathcal{L}(H_1, H_2)$. Then $\|T^\star \circ T\| = \|T \circ T^\star\| = \|T\|^2$.

(e) $T^\star$ is an isomorphism if and only if T is an isomorphism.

(f) $T^\star$ is compact if and only if T is compact.

Definition 85. Let A be an algebra over $\mathbb{K}$. The mapping $^\star\colon A \to A$ is called an *algebra involution* if it has the following properties:

- $(x+y)^\star = x^\star + y^\star$ for every $x, y \in A$,
- $(\lambda x)^\star = \overline{\lambda} x^\star$ for every $x \in A$ and $\lambda \in \mathbb{K}$,
- $(xy)^\star = y^\star x^\star$ for every $x, y \in A$,
- $(x^\star)^\star = x$ for every $x \in A$ (i.e. the mapping * is an involution).

An algebra on which there is an algebra involution is called an *algebra with an involution*.

Fact 86. *Let A be an algebra with an involution. Then $(a, \alpha)^\star = (a^\star, \overline{\alpha})$ for $(a, \alpha) \in A_{\mathrm{e}}$ is an algebra involution on A_{e} that extends the involution from A.*

Proposition 87. *Let A be an algebra with an involution and $x \in A$. Then the following hold:*

(a) If e is a left or right unit in A, then e is a unit and $e^\star = e$.

(b) Suppose A has a unit. Then $x^\star \in A^\times$ if and only if $x \in A^\times$. In this case $(x^\star)^{-1} = (x^{-1})^\star$.

(c) $\lambda \in \sigma(x)$ if and only if $\overline{\lambda} \in \sigma(x^\star)$. Therefore $r(x^\star) = r(x)$.

Proposition 88. *Let A be a commutative semi-simple Banach algebra. Then every algebra involution on A is continuous.*

Definition 89. Let A be an algebra with an involution. An element $x \in A$ is called *self-adjoint* if $x^\star = x$.

Fact 90. *Let A be an algebra with an involution and $x, y \in A$. Then the following hold:*

(a) The elements $x + x^\star$, $x^\star x$, $xx^\star$, and in the complex case also $i(x - x^\star)$ are self-adjoint.

(b) If x is self-adjoint, then also tx is self-adjoint for every $t \in \mathbb{R}$.

(c) If A is complex, then there exist unique self-adjoint elements $u, v \in A$ such that $x = u + iv$. Then $x^\star = u - iv$.

(d) If x, y are self-adjoint and commute, then xy is self-adjoint.

(e) If x is self-adjoint, then $yxy^\star$ is self-adjoint.

Definition 91. A Banach algebra A with an involution is called a *$B^\star$-algebra* if

$$\|x^\star x\| = \|x\|^2$$

for every $x \in A$.

Lemma 92. *Let A be a normed algebra with an involution. Then the following statements are equivalent:*

(i) $\|x^\star x\| \geq \|x\|^2$ for every $x \in A$.

(ii) $\|xx^\star\| \geq \|x\|^2$ for every $x \in A$.

(iii) $\|x^\star x\| = \|x\|^2$ for every $x \in A$.

(iv) $\|xx^\star\| = \|x\|^2$ for every $x \in A$.

In all cases then $\|x^\star\| = \|x\|$ for every $x \in A$.

Proposition 93. *Let A be a $B^\star$-algebra without a unit. Then there exists a norm $|||\cdot|||$ on A_{e} with the involution from Fact 86 extending the original norm on A (and equivalent to the norm from Proposition 9) such that A_{e} is a $B^\star$-algebra.*

Definition 94. Let A be an algebra with an involution.

- If A has a unit, then an element $x \in A$ is called *unitary* if $x^\star x = xx^\star = e$, or in other words $x^{-1} = x^\star$.

- An element $x \in A$ is called *normal* if it commutes with $x^\star$, i.e. if $x^\star x = xx^\star$.

Fact 95. *Let A be an algebra over $\mathbb{K}$ with an involution and $x, y \in A$.*

(a) If A has a unit and if x, y are unitary, then xy is unitary.

(b) If x is normal, then x^n is normal for every $n \in \mathbb{N}$.

(c) If A has a unit and if x is normal and y is unitary, then $yxy^\star$ is normal.

(d) If A has a unit and if x is normal and $\lambda \in \mathbb{K}$, then $\lambda e - x$ is normal.

Theorem 96. *Let A be a $B^\star$-algebra and $x \in A$.*

(a) If x is normal, then $\|x^n\| = \|x\|^n$ for every $n \in \mathbb{N}$ and if A is complex, then $r(x) = \|x\|$.

(b) If A is complex, then $r(x^\star x) = r(xx^\star) = \|x\|^2$.

(c) If A has a unit and x is unitary, then $\sigma(x) \subset \{\lambda \in \mathbb{K};\ |\lambda| = 1\}$. If moreover A is non-trivial, then $\|x\| = 1$.

(d) If x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.

Corollary 97. *Let A be a non-trivial complex commutative $B^\star$-algebra. Then $\Delta(A) \neq \emptyset$.*

Corollary 98. *Let A be a complex algebra with an involution. Then there exists at most one norm on A with which A is a $B^\star$-algebra.*

Definition 99. Let A and B be algebras with an involution. Then an algebra homomorphism $\Phi\colon A \to B$ is called a *-*homomorphism* if it preserves the operation *, i.e. if $\Phi(x^\star) = \Phi(x)^\star$ for every $x \in A$.

Corollary 100. *Let A be a complex $B^\star$-algebra. Then every multiplicative linear functional on A is a *-homomorphism.*

Corollary 101. *Let A, B be complex $B^\star$-algebras and $\Phi\colon A \to B$ a *-homomorphism. Then Φ is automatically continuous and moreover $\|\Phi\| \leq 1$.*

Corollary 102. *Let A be a complex $B^\star$-algebra and B its $B^\star$-subalgebra. If A and B has a common unit, then $B^\times = A^\times \cap B$. Further, let $x \in B$. If B has a unit which is not a unit in A, then $\sigma_A(x) = \sigma_B(x) \cup \{0\}$, in the other cases $\sigma_A(x) = \sigma_B(x)$.*

Theorem 103 (I. M. Gelfand a M. A. Naĭmark (1943))**.** *Let A be a complex commutative $B^\star$-algebra. Then the Gelfand transform is an isometric *-isomorphism of A onto $C_0(\Delta(A))$.*

Corollary 104. *A complex commutative $B^\star$-algebra A has a unit if and only if $\Delta(A)$ is compact.*

Corollary 105. *Let A and B are complex commutative $B^\star$-algebras. Then the following statements are equivalent:*

(i) A and B are isometrically *-isomorphic.

(ii) A and B are algebraically isomorphic.

(iii) The spaces $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

Theorem 106 (I. M. Gelfand a M. A. Naĭmark (1943), I. Kaplansky (1953))**.** *Every complex $B^\star$-algebra can be embedded by an isometric *-isomorphism into $\mathcal{L}(H)$ for some suitable complex Hilbert space H.*

7. Continuous calculus for normal elements of B*-algebras

Proposition 107. *Let A be a normed algebra over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f, g\colon \Omega \to A$, and $t \in \Omega$. If $f'(t)$ and $g'(t)$ exist, then $(fg)'(t) = f'(t)g(t) + f(t)g'(t)$.*

Let A be a (real) Banach algebra with a unit and $x \in A$. Then we define

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Theorem 108. *Let A be a Banach algebra over $\mathbb{K}$ with a unit e and $x \in A$.*

(a) If $y \in A$ commutes with x, then $\exp x \exp y = \exp(x + y)$.

(b) $\exp x \in A^\times$ *and* $(\exp x)^{-1} = \exp(-x)$.

(c) *Put* $f(\lambda) = \exp(\lambda x)$ *for* $\lambda \in \mathbb{K}$. *Then* $f'(\lambda) = \exp(\lambda x)x$ *for every* $\lambda \in \mathbb{K}$.

(d) *If* A *is an algebra with a continuous involution, then* $(\exp x)^\star = \exp x^\star$.

(e) *If* A *is a complex algebra with a continuous involution and* x *is self-adjoint, then* $\exp(ix)$ *is unitary.*

Theorem 109 (Bent Fuglede (1950), Calvin R. Putnam (1951))**.** *Let* A *be a complex* $B^\star$*-algebra,* $x \in A$*, and let* $a, b \in A$ *be normal and such that* $ax = xb$*. Then* $a^\star x = xb^\star$.

Definition 110. Let A be an algebra and $M \subset A$. The set

$$\operatorname{alg} M = \bigcap\{B \supset M;\ B \text{ is a subalgebra of } A\}$$

is called *algebra hull* of M.

Proposition 111. *Let* A *be an algebra and* $M \subset A$*. Then*

$$\operatorname{alg} M = \operatorname{span}\{x_1x_2\cdots x_n;\ x_1,\dots,x_n \in M, n \in \mathbb{N}\}.$$

Definition 112. Let A be a normed algebra and $M \subset A$. Then we define a *closed algebra hull* of M as

$$\overline{\operatorname{alg}}\, M = \bigcap\{B \supset M;\ B \text{ is a closed subalgebra of } A\}.$$

Proposition 113. *Let* A *be a normed algebra and* $M \subset A$*. Then* $\overline{\operatorname{alg}}\, M = \overline{\operatorname{alg} M}$.

Fact 114. *Let* A*,* B *be algebras and* $M \subset A$*. Then every algebra homomorphism* $\Phi\colon \operatorname{alg} M \to B$ *is uniquely determined by its values on* M*. If* A*,* B *are normed algebras, then every continuous algebra homomorphism* $\Phi\colon \overline{\operatorname{alg}}\, M \to B$ *is uniquely determined by its values on* M.

Proposition 115. *Let* A *be a* $B^\star$*-algebra and suppose that* $M \subset A$ *commutes and is closed under the involution. Then* $\overline{\operatorname{alg}}\, M$ *is a commutative* $B^\star$*-subalgebra of* A.

Theorem 116. *Let* A *be an algebra over* $\mathbb{K}$ *with a unit and* $x \in A$*. Let* $\Omega_2 \subset \mathbb{K}$ *be closed and* $\Omega_1 \subset \Omega_2$*. Let* $\Phi_i: C(\Omega_i) \to A$ *be an algebra homomorphism such that* $\Phi_i(1) = e$*,* $\Phi_i(Id) = x$*, in the complex case moreover* $\Phi_1(\overline{Id}) = \Phi_2(\overline{Id})$*, and let* Φ_i *be sequentially continuous from the topology of locally uniform convergence on* $C(\Omega_i)$ *to some Hausdorff topology* τ *on* A*,* $i = 1, 2$*. Then* $\Phi_1(f\restriction_{\Omega_1}) = \Phi_2(f)$ *for every* $f \in C(\Omega_2)$.

Let A be a complex B*-algebra with a unit and let $x \in A$ be normal. Set $B = \overline{\operatorname{alg}}\{e, x, x^\star\}$. Then we can define

$$f(x) = \Gamma_B^{-1}(f \circ \Gamma_B(x)). \tag{1}$$

Theorem 118 (continuous calculus)**.** *Let* A *be a complex* $B^\star$*-algebra with a unit, let* $x \in A$ *be normal and* $f \in C(\sigma(x))$*. The mapping* $\Phi\colon C(\sigma(x)) \to A$*, where* $\Phi(g) = g(x)$ *is given by the formula* (1)*, has the following properties:*

(a) Φ *is an isometric* **-isomorphism of* $C(\sigma(x))$ *onto* $B = \overline{\operatorname{alg}}\{e, x, x^\star\}$*, for which moreover* $\Phi(1) = e$ *and* $\Phi(Id) = x$.

(b) $f(x) \in A^\times$ *if and only if* $f(\lambda) \neq 0$ *for every* $\lambda \in \sigma(x)$*. In this case* $f(x)^{-1} = \frac{1}{f}(x)$.

(c) $f(x)$ *is self-adjoint if and only if* f *is real.*

(d) $\sigma(f(x)) = f(\sigma(x))$ *(spectral mapping theorem).*

(e) *If* $\Psi\colon C(\sigma(x)) \to A$ *is a* **-homomorphism for which* $\Psi(1) = e$ *and* $\Psi(Id) = x$*, then* $\Psi = \Phi$.

(f) *If* $C \subset A$ *is a commutative* $B^\star$*-subalgebra containing* e *and* x*, then* $\Gamma_C^{-1}(f \circ \Gamma_C(x)) = f(x)$.

(g) *If* $g \in C\big(f(\sigma(x))\big)$*, then* $(g \circ f)(x) = g(f(x))$.

(h) *If* $g \in H(\Omega)$*, where* $\Omega \subset \mathbb{C}$ *is an open neighbourhood of* $\sigma(x)$*, then* $\Phi(g\restriction_{\sigma(x)}) = \Psi(g)$*, where* Ψ *is the holomorphic calculus from Theorem 50.*

(i) *If* $y \in A$ *commutes with* x*, then* y *commutes also with* $f(x)$.

(j) *If* D *is a complex* $B^\star$*-algebra and* $\Theta\colon A \to D$ *is a* **-homomorphism such that* $\Theta(e)$ *is a unit in* D*, then* $f(\Theta(x)) = \Theta(f(x))$*. In particular, if* $u \in A$ *is unitary, then* $f(uxu^\star) = uf(x)u^\star$.

(k) *If* $0 \in \sigma(x)$ *and* $f(0) = 0$*, then* $f(x) \in \overline{\operatorname{alg}}\{x, x^\star\}$.

If A *does not have a unit, then we carry out the whole construction in* A_{e}*. If* $f \in C(\sigma(x))$ *is such that* $f(0) = 0$*, then* $f(x) \in A$.

Theorem 119. *Let* A *be a complex* $B^\star$*-algebra and* $x \in A$.

(a) *The element* x *is self-adjoint if and only if it is normal and* $\sigma(x) \subset \mathbb{R}$.

(b) *If* A *has a unit, then* x *is unitary if and only if it is normal and* $\sigma(x) \subset \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$.

8. Non-negative elements of B*-algebras

Definition 120. Let A be an algebra with an involution and let $x \in A$ be self-adjoint. We say that x is *non-negative*, if $\sigma(x) \subset [0, +\infty)$.

Fact 121. *An element x of a complex $B^\star$-algebra is non-negative, if and only if it is normal and $\sigma(x) \subset [0, +\infty)$.*

Proposition 122. *Let A be an algebra with an involution and let $x, y \in A$ be non-negative.*

(a) If $t \geq 0$, then tx is non-negative.

(b) If A is a complex $B^\star$-algebra, then $x + y$ is non-negative.

(c) If A is a complex Banach algebra and x and y commute, then xy is non-negative.

Fact 123. *Let A be a complex $B^\star$-algebra and $x \in A$.*

(a) If x is non-negative, then $|x| = x$.

(b) If x is self-adjoint, then $|x|^2 = x^2$.

(c) If x is non-negative, then $(\sqrt{x})^2 = x$. Moreover, $\sqrt{x}$ is the only non-negative $y \in A$ satisfying $y^2 = x$.

(d) If x is self-adjoint, then $\sqrt{x^2} = |x|$.

Proposition 124. *Let A be a complex $B^\star$-algebra. Then for every self-adjoint element $x \in A$ there exists a unique pair of non-negative elements $x^+, x^- \in A$ such that $x = x^+ - x^-$ and $x^- x^+ = x^+ x^- = 0$. Moreover, $x^+ + x^- = |x|$.*

Theorem 125 (I. Kaplansky (1953))**.** *Let A be a complex $B^\star$-algebra and $x \in A$. Then $x^\star x$ and $x x^\star$ are non-negative.*

Theorem 126 (polar decomposition)**.** *Let A be a complex $B^\star$-algebra with a unit and let $x \in A$ be invertible. Then there exist a unitary $u \in A$ and a non-negative $a \in A$ satisfying $x = ua$. This decomposition is unique.*

II. Continuous linear operators on Hilbert spaces

1. Basic properties

Theorem 127. *If H_1, H_2 are Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$, then*

(a) $\operatorname{Ker} T^\star = (\operatorname{Rng} T)^\perp$,

(b) $\operatorname{Ker} T = (\operatorname{Rng} T^\star)^\perp$,

(c) $\overline{\operatorname{Rng} T} = (\operatorname{Ker} T^\star)^\perp$,

(d) $\overline{\operatorname{Rng} T^\star} = (\operatorname{Ker} T)^\perp$.

Definition 128. Let X, Y, and Z be vector spaces over $\mathbb{K}$. A mapping $B\colon X \times Y \to Z$ is called *bilinear* if it is linear separately in the first and in the second coordinate, i.e. the mapping $x \mapsto B(x, y)$ is linear for every $y \in Y$ and $y \mapsto B(x, y)$ is linear for every $x \in X$. The mapping B is called *sesquilinear*, if it is linear in the first coordinate and conjugate-linear in the second coordinate. If $Z = \mathbb{K}$, then B is called *bilinear*, resp. *sesquilinear form*.

Proposition 129 (polarisation formula)**.** *Let X, Y be vector spaces over $\mathbb{K}$ and let $S\colon X \times X \to Y$ be a sesquilinear mapping. Then*

$$S(x, y) + S(y, x) = \frac{1}{2}\big(S(x + y, x + y) - S(x - y, x - y)\big)$$

for every $x, y \in X$. If $\mathbb{K} = \mathbb{C}$, then

$$S(x, y) = \frac{1}{4}\big(S(x + y, x + y) - S(x - y, x - y) + iS(x + iy, x + iy) - iS(x - iy, x - iy)\big)$$

for every $x, y \in X$.

Theorem 130. *Let X be an inner-product space and let $T\colon X \to X$ be a linear operator. Suppose moreover that at least one of the following condition holds:*

- *X is complex.*
- *X is a Hilbert space and T is continuous and self-adjoint.*

If $\langle Tx, x\rangle = 0$ *for every* $x \in X$*, then* $T = 0$.

Corollary 131. *Let* X *be an inner-product space and let* $S, T : X \to X$ *be linear operators. Suppose moreover that at least one of the following condition holds:*

- X *is complex.*
- X *is a Hilbert space and* S*,* T *are continuous and self-adjoint.*

If $\langle Sx, x\rangle = \langle Tx, x\rangle$ *for every* $x \in X$*, then* $S = T$.

Definition 132. Let X, Y, Z be normed linear spaces and let $B : X \times Y \to Z$ be a bilinear, resp. sesquilinear mapping. We say that B is *bounded* if $\sup_{x\in B_X, y\in B_Y} \|B(x, y)\| < +\infty$. In this case we define $\|B\| = \sup_{x\in B_X, y\in B_Y} \|B(x, y)\|$.

Proposition 133. *Let* H *be a Hilbert space. If* S *is a bounded sesquilinear form on* H*, then there exists a unique* $T \in \mathcal{L}(H)$ *such that* $S(x, y) = \langle Tx, y\rangle$ *for all* $x, y \in H$*. Moreover,* $\|T\| = \|S\|$.

Fact 134. *Let* H_1*,* H_2 *be Hilbert spaces and* $T \in \mathcal{L}(H_1, H_2)$*. Then* $\operatorname{Ker} T^\star \circ T = \operatorname{Ker} T$.

Theorem 135. *Let* H *be a Hilbert space and* $T \in \mathcal{L}(H)$*. Then the following statements are equivalent:*

(i) T *is normal.*

(ii) $\langle T^\star x, T^\star y\rangle = \langle Tx, Ty\rangle$ *for every* $x, y \in H$.

(iii) $\|T^\star x\| = \|Tx\|$ *for every* $x \in H$.

Definition 136. Let X be a normed linear spacer over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. A number $\lambda \in \mathbb{K}$ is called an *approximate eigenvalue* of the operator T if there exists a sequence $\{x_n\} \subset S_X$ such that $(\lambda I - T)x_n \to 0$. The set of all approximate eigenvalues of the operator T is called an *approximate point spectrum* of the operator T and it is denoted by $\sigma_{\text{ap}}(T)$.

Fact 137. *Let* X *be a normed linear space over* $\mathbb{K}$ *and* $T \in \mathcal{L}(X)$*. Then* $\lambda \in \mathbb{K}$ *is an approximate eigenvalue of* T *if and only if* $\lambda I - T$ *is not an isomorphism into.*

Proposition 138. *Let* X*,* Y *be normed linear spaces,* $T \in \mathcal{L}(X)$*, and let* $S : X \to Y$ *be a linear isomorphism. Then* $\sigma_{\text{ap}}(S \circ T \circ S^{-1}) = \sigma_{\text{ap}}(T)$*, where* $S \circ T \circ S^{-1} \in \mathcal{L}(Y)$.

Definition 139. Let X be an inner-product space and $T \in \mathcal{L}(X)$. The set $N_T = \{\langle Tx, x\rangle;\ x \in S_X\}$ is called a *numerical range* of the operator T.

Fact 140. *Let* X *be a normed linear space with* $\dim X_{\mathbb{R}} \neq 1$ *(i.e.* X *is either complex, or real of dimension not equal to* 1*). Then* S_X *is pathwise connected.*

Proposition 141. *Let* X *be an inner-product space over* $\mathbb{K}$ *and* $T \in \mathcal{L}(X)$.

(a) $N_{\alpha I+\beta T} = \alpha + \beta N_T$ *for any* $\alpha, \beta \in \mathbb{K}$.

(b) The set N_T *is pathwise connected.*

(c) $\sigma_{\text{p}}(T) \subset N_T \subset B_{\mathbb{K}}(0, \|T\|)$.

(d) $\sigma_{\text{ap}}(T) \subset \overline{N_T}$*. If* X *is a Hilbert space, then* $\sigma(T) \setminus \sigma_{\text{ap}}(T) \subset N_T$*, and so* $\sigma(T) \subset \overline{N_T}$.

Theorem 142. *Let* H *be a Hilbert space and let* $T \in \mathcal{L}(H)$ *be normal. Then the following hold:*

(a) $\operatorname{Ker} T = \operatorname{Ker} T^\star$.

(b) $\operatorname{Rng} T$ *is dense in* H *if and only if* T *is one-to-one.*

(c) T *is invertible if and only if there exists* $c > 0$ *such that* $\|Tx\| \geq c\|x\|$ *for every* $x \in H$.

(d) $\sigma(T) = \sigma_{\text{ap}}(T)$.

(e) $\lambda \in \sigma_{\text{p}}(T)$ *if and only if* $\overline{\lambda} \in \sigma_{\text{p}}(T^\star)$*. The eigenspace of* T *corresponding to an eigenvalue* λ *is equal to the eigenspace of* $T^\star$ *corresponding to the eigenvalue* $\overline{\lambda}$.

(f) If λ_1, λ_2 *are different eigenvalues of* T*, then* $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$.

Theorem 143. *Let* H *be a Hilbert space and* $T \in \mathcal{L}(H)$*. Then* T *is self-adjoint if and only if* $\langle Tx, y\rangle = \langle x, Ty\rangle$ *for every* $x, y \in H$*. For* T *self-adjoint the following holds:*

(a) $\langle Tx, x\rangle \in \mathbb{R}$ *for every* $x \in H$.

(b) $N_T \subset \mathbb{R}$. *If H is non-trivial and if we denote $m_T = \inf N_T$, $M_T = \sup N_T$, then $\|T\| = \max\{|m_T|, |M_T|\}$ and $\{m_T, M_T\} \subset \sigma(T) \subset [m_T, M_T]$, and so the number $\|T\|$ or $-\|T\|$ lies in $\sigma(T)$.*

(c) $r(T) = \sup\{|\lambda|;\ \lambda \in N_T\} = \|T\|$.

Proposition 144. *Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is self-adjoint if and only if $N_T \subset \mathbb{R}$.*

Corollary 145. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. If T is self-adjoint, then $\sigma(T) \subset [0, +\infty)$ if and only if $\langle Tx, x\rangle \geq 0$ for every $x \in H$. If H is complex, then T is non-negative (element of the algebra $\mathcal{L}(H)$) if and only if $\langle Tx, x\rangle \geq 0$ for every $x \in H$.*

Theorem 146. *Let H be a Hilbert space and let $P \in \mathcal{L}(H)$ be a projection. Then the following statements are equivalent:*

(i) *P is self-adjoint.*

(ii) *P is normal.*

(iii) *P is orthogonal.*

(iv) *P is non-negative.*

Lemma 147. *Let H be a Hilbert space, $S, T \in \mathcal{L}(H)$ and assume that S is self-adjoint. Then* $\operatorname{Rng} S \perp \operatorname{Rng} T$ *if and only if $ST = 0$.*

Definition 148. Let H_1, H_2 be Hilbert spaces. An operator $T \in \mathcal{L}(H_1, H_2)$ is called *unitary* if $T^\star \circ T = I_{H_1}$ and $T \circ T^\star = I_{H_2}$, or in other words $T^{-1} = T^\star$.

Theorem 149. *Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then the following statements are equivalent:*

(i) *T is unitary.*

(ii) *T is onto and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for every $x, y \in H$.*

(iii) *T is an isometry onto.*

Lemma 150. *Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Let Y be a closed subspace of H_2 such that* $\operatorname{Rng} T \subset Y$ *and let $S \in \mathcal{L}(H_1, Y)$ be defined as $Sx = Tx$ for $x \in H_1$. Then $S^\star = T^\star\restriction_Y$.*

Theorem 151. *Let H be a Hilbert space. Then $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$.*

Definition 152. Let A be a set and let $f\colon A \to A$ be a mapping. A set $B \subset A$ is called *invariant* with respect to f if $f(B) \subset B$, i.e. $f\restriction_B\colon B \to B$.

Fact 153. *Let H be a Hilbert space, $T \in \mathcal{L}(H)$, and let $M \subset H$ be a set of eigenvectors of T (not necessarily all).*

(a) *If $Y \subset H$ is invariant with respect to T, then $Y^\perp$ is invariant with respect to $T^\star$.*

(b) *$\overline{\operatorname{span}}\, M$ is invariant with respect to T.*

(c) *If T normal, then both $\overline{\operatorname{span}}\, M$ and $(\overline{\operatorname{span}}\, M)^\perp$ are invariant with respect to both T and $T^\star$.*

(d) *Let $Y \subset H$ be a closed subspace invariant with respect to both T and $T^\star$. Then $(T\restriction_Y)^\star = T^\star\restriction_Y$. So if T is self-adjoint, resp. normal, then $T\restriction_Y \in \mathcal{L}(Y)$ is self-adjoint, resp. normal.*

Theorem 154 (spectral decomposition of a normal compact operator; D. Hilbert (1904), Erhard Schmidt (1907))**.** *Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Suppose further that*

- *T is self-adjoint or*
- *H is complex and T is normal.*

Then there exist an orthonormal basis B of H consisting of eigenvectors of T. The set of all vectors from B corresponding to non-zero eigenvalues of T is countable and if we enumerate it by an arbitrary injective sequence $\{e_n\}_{n=1}^N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, then $\{e_n\}$ is an orthonormal basis of $\overline{\operatorname{Rng} T}$ and

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n$$

for every $x \in H$, where λ_n is the eigenvalue corresponding to the eigenvector e_n.

If $\{\lambda_n\}_{n=1}^M$, $M \in \mathbb{N}_0 \cup \{\infty\}$ *is an injective sequence of all eigenvalues of* T *and* P_n *is the orthogonal projection onto* $\operatorname{Ker}(\lambda_n I - T)$, *then*

$$I = \sum_{n=1}^{M} P_n,$$

where the series converges pointwise unconditionally (i.e. $x = \sum_{n=1}^M P_n x$ *unconditionally for every* $x \in H$*) and*

$$T = \sum_{n=1}^{M} \lambda_n P_n,$$

where the series converges unconditionally in the space $\mathcal{L}(H)$.

Theorem 155 (representation of a compact operator; E. Schmidt (1907))**.** *Let* H_1, H_2 *be Hilbert spaces and* $T \in \mathcal{K}(H_1, H_2)$. *Then there exist* $N \in \mathbb{N}_0 \cup \{\infty\}$, *a sequence of positive numbers* $\{\lambda_n\}_{n=1}^N$, *and orthonormal systems* $\{u_n\}_{n=1}^N \subset H_1$ *and* $\{v_n\}_{n=1}^N \subset H_2$ *such that*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n$$

for every $x \in H$. *Further,* $\{\lambda_n^2\}_{n=1}^N$ *is a sequence of all non-zero eigenvalues of the operator* $T^\star \circ T$, *and for every* $\lambda > 0$ *the number of elements of the set* $\{n \in \mathbb{N};\ \lambda_n^2 = \lambda\}$ *is equal to* $\dim \operatorname{Ker}(\lambda I - T^\star \circ T)$. *So the sequence* $\{\lambda_n\}_{n=1}^N$ *is determined uniquely up to a permutation and if* $N = \infty$, *then* $\lambda_n \to 0$.

2. Bounded Borel calculus

Definition 156. Let X, Y be normed linear spaces. We define the following locally convex topologies on the space $\mathcal{L}(X, Y)$:

- the *strong operator topology* τ_{SOT} is generated by the system of seminorms $\{p_x(T) = \|Tx\|;\ x \in X\}$,
- the *weak operator topology* τ_{WOT} is generated by the system of seminorms $\{p_{x,f}(T) = |f(Tx)|;\ x \in X, f \in Y^*\}$.

The symbol $\mathrm{Bf_b}(X)$ denotes the set of all bounded Borel functions on a topological space X.

Definition 157. Let X be a Banach space over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. We say that a mapping $\Psi\colon \mathrm{Bf_b}(\sigma(T)) \to \mathcal{L}(X)$ is a *Borel functional calculus* for T if Ψ is an algebra homomorphism, $\Psi(1) = I$, $\Psi(Id) = T$, and if $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to $f \in \mathrm{Bf_b}(\sigma(T))$, then $\Psi(f_n) \to \Psi(f)$ in the topology τ_{WOT}.

Let A be an algebra over $\mathbb{K}$ with a unit, τ a Hausdorff topology on A, $x, y \in A$, and $F \subset \mathbb{K}$ closed. A homomorphism $\Phi\colon \mathrm{Bf_b}(F) \to A$ will be called a Borel calculus on F for τ and a pair (x, y) if $\Phi(1) = e$, $\Phi(Id) = x$, $\Phi(\overline{Id}) = y$, and $\Psi(f_n) \xrightarrow{\tau} \Psi(f)$ whenever $\{f_n\} \subset \mathrm{Bf_b}(F)$ is a bounded sequence converging pointwise to $f \in \mathrm{Bf_b}(F)$.

Theorem 158. *Let* A *be a Banach algebra over* $\mathbb{K}$ *with a unit,* τ *a Hausdorff topology on* A *(non-strictly) weaker than norm, and* $x, y \in A$. *Assume that there exists a Borel calculus* Ψ *on a closed* $F \subset \mathbb{K}$ *for* τ *and a pair* (x, y). *Then there is a Borel calculus* Φ *on* $\sigma(x)$ *for* τ *and a pair* (x, y). *If moreover* Ψ_1 *is a Borel calculus on* F_1 *for* τ *and a pair* (x, y), *then* $\Psi_1(f) = \Phi(f\restriction_{\sigma(x)})$ *for every* $f \in \mathrm{Bf_b}(F_1)$.

Lemma 159. *Let* H *be a Hilbert space and* $\{x_n\}_{n=1}^\infty \subset H$. *If* $x_n \to x \in H$ *weakly and* $\|x_n\| \to \|x\|$, *then* $x_n \to x$ *(in the norm).*

Let H be a complex Hilbert space and let $T \in \mathcal{L}(H)$ be a normal operator. For fixed $x, y \in H$ consider the function $\varphi_{x,y}\colon C(\sigma(T)) \to \mathbb{C}$ defined by

$$\varphi_{x,y}(f) = \langle f(T)x, y \rangle.$$

There exist a regular Borel complex measure $\mu_{x,y}$ on $\sigma(T)$ such that

$$\varphi_{x,y}(f) = \int_{\sigma(T)} f \,\mathrm{d}\mu_{x,y}$$

for every $f \in C(\sigma(T))$, and $\|\mu_{x,y}\| = \|\varphi_{x,y}\| \le \|x\|\|y\|$.

For $f \in \mathrm{Bf_b}(\sigma(T))$ there exist a unique operator $f(T) \in \mathcal{L}(H)$ such that

$$\langle f(T)x, y \rangle = \int_{\sigma(T)} f \,\mathrm{d}\mu_{x,y} \tag{2}$$

for every $x, y \in H$. Moreover, $\|f(T)\| \le \|f\|_\infty$.

Theorem 160. *Let H be a complex Hilbert space, let $T \in \mathcal{L}(H)$ be a normal operator and $f \in \mathrm{Bf_b}(\sigma(T))$. The mapping $\Phi\colon \mathrm{Bf_b}(\sigma(T)) \to \mathcal{L}(H)$, where $\Phi(g) = g(T)$ is defined above, is a Borel functional calculus for T with the following properties:*

(a) Φ is a *-homomorphism and if we denote by Ψ the continuous calculus for T from Theorem 118, then $\Phi\restriction_{C(\sigma(T))} = \Psi$. If H is non-trivial, then $\|\Phi\| = 1$.

(b) If $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to f, then $\Phi(f_n) \to \Phi(f)$ in the topology τ_{SOT}.

(c) If Ψ is a Borel functional calculus for T which is moreover a *-homomorphism, then $\Psi(g) = \Phi(g)$ for every $g \in \mathrm{Bf_b}(\sigma(T))$.

(d) $f(T)$ is normal. If f is real, then $f(T)$ is self-adjoint. If $|f| = 1$, then $f(T)$ is unitary.

(e) $\sigma(f(T)) \subset \overline{f(\sigma(T))}$.

(f) If $g \in \mathrm{Bf_b}\big(\overline{\mathrm{Rng}\, f}\big)$, then $(g \circ f)(T) = g(f(T))$.

(g) If $S \in \mathcal{L}(H)$ commutes with T, then S commutes also with $f(T)$.

(h) If $U \in \mathcal{L}(H)$ is unitary, then $f(UTU^\star) = Uf(T)U^\star$.

3. Polar decomposition

Theorem 161 (polar decomposition)**.** *Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is normal if and only if there exist a unitary $U \in \mathcal{L}(H)$ and a non-negative $A \in \mathcal{L}(H)$ such that $T = UA = AU$. This decomposition is unique if and only if T is one-to one.*

Corollary 162. *Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is normal if and only if there exists a unitary $U \in \mathcal{L}(H)$ such that $T^\star = UT = TU$.*

Theorem 163. *Let H_1, H_2 be complex Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then there exists a unique pair of operators $A \in \mathcal{L}(H_1)$ and $U \in \mathcal{L}(\overline{\mathrm{Rng}\, A}, \overline{\mathrm{Rng}\, T})$ such that $T = U \circ A$, A is non-negative, and U is unitary. If T is an isomorphism, then A is an automorphism of H_1.*

Proposition 164. *Let $T \in \mathcal{L}(\mathbb{C}^n)$. Then there exist a unitary $U \in \mathcal{L}(\mathbb{C}^n)$ and a non-negative $A \in \mathcal{L}(\mathbb{C}^n)$ such that $T = UA$.*

4. Spectral decomposition of an operator

Definition 165. Let $\mathcal{S}$ be a σ-algebra and X a topological vector space. A mapping $\mu\colon \mathcal{S} \to X$ is called a *vector measure* if $\mu\big(\bigcup_{n=1}^\infty A_n\big) = \sum_{n=1}^\infty \mu(A_n)$ for every sequence $\{A_n\}_{n=1}^\infty$ of pairwise disjoint sets from $\mathcal{S}$.

Fact 166. *Let X, Y be topological vector spaces, $\mu\colon \mathcal{S} \to X$ a vector measure, and $T\colon X \to Y$ a continuous linear mapping. Then $T \circ \mu$ is also a vector measure.*

Proposition 167. *Let X, Y be normed linear spaces over $\mathbb{K}$, $\mathcal{S}$ a σ-algebra, and $\mu\colon \mathcal{S} \to (\mathcal{L}(X, Y), \tau_{\mathrm{WOT}})$ a vector measure. Then for every $x \in X$ and $f \in Y^*$ the function $\mu_{x,f}\colon \mathcal{S} \to \mathbb{K}$ given by*

$$\mu_{x,f}(A) = f(\mu(A)x)$$

is a complex measure on $\mathcal{S}$. The mapping $B\colon (x, f) \mapsto \mu_{x,f}$ is a bilinear mapping from $X \times Y^*$ to a normed linear space of complex measures on $\mathcal{S}$. If moreover X is a Banach space, then $\sup_{A\in\mathcal{S}} \|\mu(A)\| < +\infty$ and B is bounded.

Theorem 168 (B. J. Pettis (1938))**.** *Let X be a normed linear space and $\mu\colon \mathcal{S} \to (X, w)$ a vector measure. Then μ is also a vector measure as a mapping into $(X, \|\cdot\|)$.*

Corollary 169. *Let X, Y be normed linear spaces, $\mathcal{S}$ a σ-algebra, and $\mu\colon \mathcal{S} \to (\mathcal{L}(X, Y), \tau_{\mathrm{WOT}})$ a vector measure. Then μ is also a vector measure as a mapping into $(\mathcal{L}(X, Y), \tau_{\mathrm{SOT}})$.*

By $\mathrm{Bs}(X)$ we denote the σ-algebra of Borel subsets of a topological space X.

Definition 170. Let X be a Banach space over $\mathbb{K}$. A *resolution of the identity* on X is a vector measure $E\colon \mathrm{Bs}(\mathbb{K}) \to (\mathcal{L}(X), \tau_{\mathrm{SOT}})$ with the following properties:

(i) $E(A)$ is a projection for every Borel $A \subset \mathbb{K}$.

(ii) $E(\mathbb{K}) = I$.

(iii) $E(A \cap B) = E(A)E(B)$ for every Borel $A, B \subset \mathbb{K}$.

If X is a Hilbert space and all projections $E(A)$ are orthogonal, then E is called an *orthogonal* resolution of the identity on X.

Fact 171. *Let X be a Banach space over $\mathbb{K}$ and E a resolution of the identity on X.*

(a) The projections $E(A)$ and $E(B)$ commute for every $A, B \in \mathrm{Bs}(\mathbb{K})$.

(b) If $A, B \in \mathrm{Bs}(\mathbb{K})$, $B \subset A$, then $\mathrm{Rng}\, E(B) \subset \mathrm{Rng}\, E(A)$ and $\mathrm{Ker}\, E(B) \supset \mathrm{Ker}\, E(A)$.

(c) If $\{A_n\} \subset \mathrm{Bs}(\mathbb{K})$, then $\bigcap_{n=1}^{\infty} \mathrm{Ker}\, E(A_n) \subset \mathrm{Ker}\, E\big(\bigcup_{n=1}^{\infty} A_n\big)$.

(d) $E_{x,f}$ is a regular Borel complex measure on $\mathbb{K}$ for every $x \in X$ a $f \in X^*$.

Let moreover X be a Hilbert space and E orthogonal.

(e) If $A, B \in \mathrm{Bs}(\mathbb{K})$ are disjoint, then $\mathrm{Rng}\, E(A) \perp \mathrm{Rng}\, E(B)$.

(f) $E_{x,x}$ is a finite regular Borel non-negative measure on $\mathbb{K}$ and $\|E_{x,x}\| = \|x\|^2$ for every $x \in X$.

Lemma 172. *Let X be a Banach space over $\mathbb{K}$ and suppose $E\colon \mathrm{Bs}(\mathbb{K}) \to \mathcal{L}(X)$ has the following properties:*

(i) $E(A)$ is a projection for every Borel $A \subset \mathbb{K}$.

(ii) $E(\mathbb{K}) = I$.

(iii) $E(A \cap B) = E(A)E(B)$ for every Borel $A, B \subset \mathbb{K}$.

(iv) $E_{x,f}\colon \mathrm{Bs}(\mathbb{K}) \to \mathbb{K}$, $E_{x,f}(A) = f(E(A)x)$ is a Borel complex measure on $\mathbb{K}$ for every $x \in X$ and $f \in X^*$.

Then E is a resolution of the identity on X.

If X is a complex Hilbert space, then instead of (iv) it suffices to assume that $E_{x,x}\colon \mathrm{Bs}(\mathbb{K}) \to \mathbb{C}$, $E_{x,x}(A) = \langle E(A)x, x\rangle$ is a finite Borel measure on $\mathbb{C}$ for every $x \in X$.

Proposition 173. *Let X, Y be Banach spaces over $\mathbb{K}$, let E be a resolution of the identity on X, and let $S\colon X \to Y$ be a linear isomorphism. Then $F\colon A \mapsto S \circ E(A) \circ S^{-1}$, $A \in \mathrm{Bs}(\mathbb{K})$ is a resolution of the identity on Y. If moreover X, Y are Hilbert spaces, S is an isometry (and so unitary), and E is orthogonal, then F is also orthogonal.*

Definition 174. Let X be a Banach space over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. We say that E is a *resolution of the identity with respect to the operator* T if E is a resolution of the identity on X such that for every Borel $A \subset \mathbb{K}$ the following holds:

(i) the projection $E(A)$ commutes with T,

(ii) if we set $T_A = T\restriction_{\mathrm{Rng}\, E(A)}$, then $\sigma(T_A) \subset \overline{A}$.

Proposition 175. *Let X be a Banach space over $\mathbb{K}$, $T \in \mathcal{L}(X)$, and E a resolution of the identity with respect to T.*

(a) $\sigma(T_A) \subset \sigma(T)$ for every Borel $A \subset \mathbb{K}$.

(b) In the complex case $E(\sigma(T)) = I$.

(c) If $E(\sigma(T)) = I$ (in particular if X is complex), then $E(G) \neq 0$ for every (relatively) open non-empty $G \subset \sigma(T)$.

(d) $\mathrm{Ker}(\lambda I - T) \subset \mathrm{Rng}\, E(\{\lambda\})$ for every $\lambda \in \mathbb{K}$. In particular, if λ is an eigenvalue of T, then $E(\{\lambda\}) \neq 0$.

Lemma 176. *Let X, Y be normed linear spaces, $T \in \mathcal{L}(X)$, let $Z \subset X$ be a subspace invariant with respect to T, and let $S\colon X \to Y$ be a linear isomorphism. Then $S(Z)$ is invariant with respect to $U = S \circ T \circ S^{-1} \in \mathcal{L}(Y)$ and $\sigma(U\restriction_{S(Z)}) = \sigma(T\restriction_Z)$.*

Proposition 177. *Let X, Y be Banach spaces over $\mathbb{K}$, $T \in \mathcal{L}(X)$, and $S\colon X \to Y$ a linear isomorphism. If E is a resolution of the identity with respect to T, then $F\colon A \mapsto S \circ E(A) \circ S^{-1}$, $A \in \mathrm{Bs}(\mathbb{K})$, is a resolution of the identity with respect to the operator $U = S \circ T \circ S^{-1} \in \mathcal{L}(Y)$.*

Theorem 178. *Let X be a Banach space over $\mathbb{K}$. If Ψ is a Borel functional calculus for $T \in \mathcal{L}(X)$, then there exists a resolution of the identity E with respect to T such that*

$$\phi(Tx) = \int_{\sigma(T)} \lambda \, \mathrm{d}E_{x,\phi}(\lambda)$$

for every $x \in X$ and $\phi \in X^*$. This resolution has the following properties:

(a) $E(A) = \Psi(\chi_{A \cap \sigma(T)})$ for every Borel $A \subset \mathbb{K}$.

(b)

$$\phi(\Psi(f)x) = \int_{\sigma(T)} f \, \mathrm{d}E_{x,\phi}$$

for every $f \in \mathrm{Bf_b}(\sigma(T))$ *and every* $x \in X$ *and* $\phi \in X^*$.

(c) $E(\{\lambda\})$ *is a projection onto* $\mathrm{Ker}(\lambda I - T)$ *for every* $\lambda \in \mathbb{K}$.

(d) $\lambda \in \sigma_\mathrm{p}(T)$ *if and only if* $E(\{\lambda\}) \neq 0$.

(e) If X *is complex and* λ *an isolated point of* $\sigma(T)$*, then* $\lambda \in \sigma_\mathrm{p}(T)$.

(f) If X *is a Hilbert space and* Ψ *is a* **-homomorphism, then* E *is orthogonal.*

On the other hand, if E *is a resolution of the identity on* X *such that* $E(K) = I$ *for some compact* $K \subset \mathbb{K}$*, then there exists a unique mapping* $\Psi\colon \mathrm{Bf_b}(K) \to \mathcal{L}(X)$ *such that (b) holds. This* Ψ *is a Borel functional calculus for* $T = \Psi(Id)$*,* E *is a resolution of the identity with respect to* T*, and (a)–(e) holds. If moreover* X *is a complex Hilbert space and* E *is orthogonal, then* Ψ *is a* **-homomorphism and* T *is normal.*

Corollary 179. *Let* H *be a complex Hilbert space and* $T \in \mathcal{L}(H)$ *a normal operator. Then there exists a unique orthogonal resolution of the identity* E *on* H *such that there is a compact* $K \subset \mathbb{C}$ *containing* $\sigma(T)$*,* $E(K) = I$*, and*

$$\langle Tx, x\rangle = \int_K \lambda \, \mathrm{d}E_{x,x}(\lambda)$$

for every $x \in H$*. This resolution is given by the formula* $E(A) = \chi_A(T)$*. It is an orthogonal resolution of the identity with respect to* T*.*

$$\langle f(T)x, y\rangle = \int_{\sigma(T)} f \, \mathrm{d}E_{x,y}$$

for every $f \in \mathrm{Bf_b}(\sigma(T))$ *and every* $x, y \in H$*. Further, (c), (d), (e) of Theorem 178 hold.*

Definition 180. Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be measurable spaces, X a topological vector space, $\mu\colon \mathcal{S} \to X$ a vector measure, and $f\colon S \to T$ a measurable mapping. The mapping $f(\mu)\colon \mathcal{T} \to X$ defined by the formula $f(\mu)(A) = \mu(f^{-1}(A))$ for $A \in \mathcal{T}$ is called an *image of the vector measure* μ.

Proposition 181. *Let* X *be a Banach space over* $\mathbb{K}$*,* E *a resolution of the identity with respect to* $T \in \mathcal{L}(X)$ *such that* $E(K) = I$ *for some compact* $K \subset \mathbb{K}$*, and* $f \in \mathrm{Bf_b}(K)$*. Then* $f(E)$ *is a resolution of the identity with respect to* $f(T) = \Psi(f)$*, where* Ψ *is the Borel functional calculus for* T *from Theorem 178.*