Functional analysis 2

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- Banach algebras

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I. Banach algebras

1. Basic properties

Definition 1

We say that $(A, +, -, 0, \cdot_{\mathrm{s}}, \cdot)$ is an algebra over $\mathbb{K}$ if $(A, +, -, 0, \cdot_{\mathrm{s}})$ is a vector space over $\mathbb{K}$, $(A, +, -, \cdot, 0)$ is a ring, and moreover $(\alpha \cdot_{\mathrm{s}} a) \cdot b = a \cdot (\alpha \cdot_{\mathrm{s}} b) = \alpha \cdot_{\mathrm{s}} (a \cdot b)$ for all $a, b \in A$ and $\alpha \in \mathbb{K}$.

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Proposition 2

Let A be an algebra over $\mathbb{K}$. Put $A_{\mathrm{e}} = A \times \mathbb{K}$ and define vector operations on A_{e} in the usual way (i.e. componentwise) and further multiplication of the elements of A_{e} by the formula

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad \text{for } a, b \in A, \alpha, \beta \in \mathbb{K}.$$

Then A_{e} is an algebra with the unit $(0,1)$ and A can be identified with its subalgebra $A \times \{0\}$. If A is commutative, then so is A_{e}.

Let A, B be algebras over $\mathbb{K}$. (Algebra) homomorphism $\Phi \colon A \to B$ is a mapping which is a homomorphism between the respective vector spaces (i.e. it is linear) and also it is a homomorphism between the respective rings (i.e. it is multiplicative, or $\Phi(ab) = \Phi(a)\Phi(b)$).

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Φ is called an (algebraic) isomorphism of algebras A and B if Φ is a bijection.

Fact 3

Let A be an algebra, B an algebra with a unit e, and $\Phi\colon A \to B$ a homomorphism. Then $\widetilde{\Phi}\colon A_{\mathrm{e}} \to B$, $\widetilde{\Phi}(x,\lambda) = \Phi(x) + \lambda e$ is a homomorphism extending Φ.

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Proposition 4

Let A be an algebra with a unit e and B a subalgebra of A not containing e. Then $C = B + \mathrm{span}\{e\}$ *is a subalgebra of A and the mapping* $\Phi\colon B_{\mathrm{e}} \to C$, $\Phi(x,\lambda) = x + \lambda e$ *is an isomorphism.*

Definition 5

A pair $(A, \|\cdot\|)$ is called a normed algebra if A is an algebra, $(A, \|\cdot\|)$ is a normed linear space, and $\|ab\| \leq \|a\| \|b\|$ for each $a, b \in A$. If the metric generated by $\|\cdot\|$ is complete, then $(A, \|\cdot\|)$ is called a Banach algebra.

Proposition 6

Let $(A, \|\cdot\|)$ be a normed algebra. The multiplication of elements of A is Lipschitz on bounded sets (and in particular continuous) as a mapping from $A \times A$ to A.

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Corollary 7

Let A *be a normed algebra and* B *a subalgebra of* A. *Then* $\overline{B}$ *is also a subalgebra of* A.

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Corollary 7

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Corollary 8

Every normed algebra A has a completion, i.e. a Banach algebra such that A is its dense subalgebra. This completion is unique up to an isometry. If A has a unit e, then e is also a unit in the completion of A.

Proposition 9

Let $(A, \|\cdot\|)$ *be a normed algebra. If we define a norm on* A_{e} *by the formula* $\|(a, \alpha)\|_{A_{\mathrm{e}}} = \|a\| + |\alpha|$ *(i.e.* $A_{\mathrm{e}} = A \oplus_1 \mathbb{K}$*), then* A_{e} *with this norm is a normed algebra. If* $(A, \|\cdot\|)$ *is a Banach algebra, then so is* A_{e} *with the norm above.*

Definition 10

Let A and B be normed algebras and $\Phi\colon A \to B$ an (algebra) homomorphism. We say that Φ is an isomorphism of normed algebras A and B (or just an isomorphism) if Φ is a homeomorphism of A onto B; we say that Φ is an isomorphism of A into B (or just an isomorphism into) if Φ is an isomorphism of A onto $\operatorname{Rng}\Phi$.

Theorem 11

Let A be a normed algebra. For each $a \in A$ we define a left translation $L_a\colon A \to A$ by the formula $L_a(x) = ax$. Then $L_a \in \mathcal{L}(A)$ and the mapping $l\colon A \to \mathcal{L}(A)$, $l(a) = L_a$ is a continuous algebra homomorphism with $\|l\| \leq 1$.

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Corollary 12

Let $(A, \|\cdot\|)$ be a non-trivial normed algebra with a unit. Then there exists an equivalent norm $|||\cdot|||$ on A such that $(A, |||\cdot|||)$ is a normed algebra and $|||e||| = 1$.

Recall that in a ring with a unit (or more generally in a monoid) the following holds: if x has a left and a right inverse, then these are equal (and it is then and inverse to x). In particular, inverses to invertible elements are uniquely determined. Further, the invertible elements form a group, i.e. if $x, y \in A$ are invertible, then also xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$. This group of invertible elements will be denoted by $A^\times$.

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Fact 13

Let A be an algebra with a unit and B its subalgebra containing e. Then $B^\times \subset A^\times \cap B$.

Fact 14

Let A, B be semigroups, $\Phi\colon A \to B$ a homomorphism onto, and let A be moreover a monoid with a unit e. Then B is a monoid with a unit $\Phi(e)$ and if $x \in A$ is invertible, then $\Phi(x)$ is invertible and $\Phi(x)^{-1} = \Phi(x^{-1})$. If moreover Φ is a bijection, then $\Phi\restriction_{A^\times}$ is an isomorphism of the groups $A^\times$ and $B^\times$.

Lemma 15

Let A be a normed algebra wit a unit and $x \in A$. If the series $\sum_{n=0}^{\infty} x^n$ converges, then $\sum_{n=0}^{\infty} x^n = (e - x)^{-1}$.

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Lemma 16

Let A be a Banach algebra with a unit.

(a) *If $x \in U_A$, then the series $\sum_{n=0}^{\infty} x^n$ converges absolutely and so $\sum_{n=0}^{\infty} x^n = (e - x)^{-1}$.*

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(b) *Let $x \in A^\times$ and let $h \in A$ be such that $\|h\| < \frac{1}{\|x^{-1}\|}$. Then $x + h \in A^\times$. If moreover $\|h\| \leq \frac{1}{2\|x^{-1}\|}$, then $\left\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}\right\| \leq 2\|x^{-1}\|^3\|h\|^2$.*

Definition 17

Let G be a group and τ a topology on G. We say that (G, τ) is a topological group if the group operations (i.e. multiplication $\cdot : G \times G \to G$ and inversion $^{-1} : G \to G$) are continuous.

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Theorem 18

Let A be a Banach algebra with a unit. Then $A^\times$ is an open subset of A and it is a topological group.

Proposition 19

Let A be a Banach algebra with a unit and B its closed subalgebra containing e. Then $(\partial_B B^\times) \cap A^\times = \emptyset$ *and*

$$B^\times = \bigcup \{C \subset B;\ C \text{ is a component of } A^\times \cap B \text{ intersecting } B^\times\}.$$

2. Spectral theory

Definition 20

Let A be an algebra with a unit. For $x \in A$ we define the resolvent set of x as

$$\rho(x) = \{\lambda \in \mathbb{K};\ \lambda e - x \in A^{\times}\}$$

and the spectrum of x as

$$\sigma(x) = \mathbb{K} \setminus \rho(x).$$

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If A does not have a unit, then for $x \in A$ we define the above notions with respect to the algebra A_{e}.

Definition 21

An element x of a groupoid is called idempotent if $x^2 = x$.

Proposition 22

Let A, B be algebras and $\Phi : A \to B$ an algebraic isomorphism. Then $\sigma(\Phi(x)) = \sigma(x)$ for every $x \in A$.

Lemma 23

Let M be a monoid and $x, y \in M$. If at least two of the elements x, y, xy, and yx are invertible, then all four are invertible.

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Let A be an algebra over $\mathbb{K}$.

(a) *If A is non-trivial, then $\sigma(0) = \{0\}$.*

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(c) *If $x \in A$, $n \in \mathbb{N}$, and $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$.*

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(d) *If $x \in A^\times$, then $\lambda \in \sigma(x)$ if and only if $\frac{1}{\lambda} \in \sigma(x^{-1})$.*

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(e) *If $x, y \in A$, then the sets $\sigma(xy)$ and $\sigma(yx)$ differ at most by the element 0. If moreover $x \in A^\times$, then $\sigma(xy) = \sigma(yx)$.*

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(f) *If $z \in A^\times$, then $\sigma(x) = \sigma(zxz^{-1})$ for every $x \in A$.*

Proposition 25

Let X, Y be normed linear spaces, $T \in \mathcal{L}(X)$, and let $S\colon X \to Y$ be a linear isomorphism. Then the operator $S \circ T \circ S^{-1} \in \mathcal{L}(Y)$ has the following property: $\sigma(S \circ T \circ S^{-1}) = \sigma(T)$ a $\sigma_{\mathrm{p}}(S \circ T \circ S^{-1}) = \sigma_{\mathrm{p}}(T)$.

Fact 26

Let A be an algebra and B an ideal in A. Then B is also an ideal in A_{e}.

Proposition 27

Let A be an algebra.

(a) $0 \in \sigma_{A_e}(x)$ *for every* $x \in A$. *So, if A does not have a unit, then* $0 \in \sigma(x)$ *for every* $x \in A$.

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(c) *Suppose that* A *has a unit* e, B *is a subalgebra of* A *not containing* e, *and* $C = B + \operatorname{span}\{e\}$. *Then* $\sigma_C(x) = \sigma_{B_e}(x)$ *for every* $x \in B$.

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(d) *Let* B *be a subalgebra of* A *and* $x \in B$. *If* B *has a unit which is not a unit in* A, *then* $\sigma_A(x) \subset \sigma_B(x) \cup \{0\}$, *in the other cases* $\sigma_A(x) \subset \sigma_B(x)$.

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(c) *Suppose that A has a unit e, B is a subalgebra of A not containing e, and* $C = B + \mathrm{span}\{e\}$. *Then* $\sigma_C(x) = \sigma_{B_e}(x)$ *for every* $x \in B$.

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(e) *If B is a proper ideal in A, then* $\sigma_{B_e}(x) = \sigma_A(x)$ *for every* $x \in B$.

Proposition 28

Let A, B be algebras, $\Phi : A \to B$ a homomorphism, and $x \in A$. If A has a unit e and $\Phi(e)$ is not a unit in B, then $\sigma_B(\Phi(x)) \subset \sigma_A(x) \cup \{0\}$, in the other cases $\sigma_B(\Phi(x)) \subset \sigma_A(x)$.

Definition 29

Let A be an algebra. For $x \in A$ we define the spectral radius of x as

$$r(x) = \sup\{|\lambda| \in [0, +\infty);\ \lambda \in \sigma(x)\}.$$

Theorem 30

Let A be a Banach algebra and $x \in A$. Then $\rho(x)$ is open, $\sigma(x)$ is compact, and

$$r(x) \leq \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_{n \to \infty} \sqrt[n]{\|x^n\|}.$$

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Lemma 31

Let $\{a_n\}$ be a sequence of real numbers.

(a) *If $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n} < +\infty$.*

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(b) *If $\{a_n\}$ is non-negative and $a_{m+n} \leq a_m a_n$ for all $m, n \in \mathbb{N}$, then* $\lim_{n\to\infty} \sqrt[n]{a_n} = \inf_{n\in\mathbb{N}} \sqrt[n]{a_n} \in \mathbb{R}$.

Theorem 32

Let A be a Banach algebra with a unit, B its closed subalgebra containing e, and $x \in B$. Then the following hold:

(a) $\partial\rho_B(x) \subset \partial\rho_A(x)$ *and*

$$\rho_B(x) = \bigcup\{C \subset \mathbb{K};\ C \text{ is a component of } \rho_A(x) \text{ intersecting } \rho_B(x)\}.$$

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(c) *If $\rho_A(x)$ is connected, then $\sigma_B(x) = \sigma_A(x)$.*

(d) *If $\sigma_B(x)$ has an empty interior, then $\sigma_B(x) = \sigma_A(x)$.*

Definition 33

Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f\colon \Omega \to Y$, and $a \in \Omega$. If $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} \in Y$ exists, then this limit is called the derivative of the mapping f at a and it is denoted by $f'(a)$.

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Fact 34

Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f\colon \Omega \to Y$, and $a \in \Omega$. If $f'(a)$ exists, then $(\phi \circ f)'(a) = \phi(f'(a))$ for every $\phi \in Y^*$.

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Let Y be a normed linear space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f\colon \Omega \to Y$, and $a \in \Omega$. If $f'(a)$ exists, then f is continuous at a.

Definition 36

Let A be an algebra over $\mathbb{K}$ with a unit. On $\rho(x)$ we define the resolvent (or the resolvent mapping) of the element x by the formula

$$R_x(\lambda) = (\lambda e - x)^{-1}, \quad \lambda \in \rho(x).$$

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If A does not have a unit, then we define the resolvent with respect to the algebra A_{e}.

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If A does not have a unit, then we define the resolvent with respect to the algebra A_{e}.

Proposition 37

Let A be a Banach algebra and $x \in A$. Then the mapping $\lambda \mapsto R_x(\lambda)$ has a derivative at every point of the set $\rho(x)$.

Definition 38

Let Y be a complex normed linear space, $\Omega \subset \mathbb{C}$ an open set, and $f\colon \Omega \to Y$. We say that f is holomorphic on Ω, if $f'(z)$ exists for every $z \in \Omega$.

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Theorem 39 (Liouville's theorem)

Let Y be a complex normed linear space and let $f\colon \mathbb{C} \to Y$ be holomorphic on $\mathbb{C}$. If f is bounded, then it is constant.

Theorem 40

Let A be a complex Banach algebra and $x \in A$.

(a) *The resolvent mapping R_x is holomorphic on $\rho(x)$.*

(b) *If A is non-trivial, then $\sigma(x) \neq \emptyset$.*

(c) $r(x) = \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_{n \to \infty} \sqrt[n]{\|x^n\|}$
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Corollary 41

If A is a complex Banach algebra, $x \in A$, and $\lambda \in \mathbb{C}$, $|\lambda| > r(x)$, then the sum $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ converges absolutely. So if A has a unit, then $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

Theorem 42 (S. Mazur (1938), I. M. Gelfand (1941))

Let A be a non-trivial complex Banach algebra with a unit. If $A^\times = A \setminus \{0\}$, then A is isomorphic to $\mathbb{C}$. If moreover $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.

3. Holomorphic calculus

3. Holomorphic calculus

Let A be a Banach algebra over $\mathbb{K}$ with a unit and $x \in A$. Further let $\mathscr{F}$ be some algebra of functions defined on a subset of $\mathbb{K}$ that contains polynomials. A functional calculus for x will be some homomorphism $\Phi \colon \mathscr{F} \to A$ such that $\Phi(1) = e$, $\Phi(Id) = x$, and which is moreover continuous, resp. sequentially continuous, in some convenient topologies on $\mathscr{F}$ and A.

Theorem 43

Let A be a complex algebra with a unit and $x \in A$. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open neighbourhoods of $\sigma(x)$ and let $\Phi_i\colon H(\Omega_i) \to A$ be an algebra homomorphism such that $\Phi_i(1) = e$, $\Phi_i(Id) = x$, and Φ_i is sequentially continuous from the topology of locally uniform convergence on $H(\Omega_i)$ to some Hausdorff topology τ on A, $i = 1, 2$. If $f_i \in H(\Omega_i)$, $i = 1, 2$ are such that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$, then $\Phi_1(f_1) = \Phi_2(f_2)$.

Let X be a complex Banach space, $\gamma\colon [a, b] \to \mathbb{C}$ a path, and $f\colon \langle\gamma\rangle \to X$ a continuous mapping. The integral of f along γ is defined by

$$\int_\gamma f = \int_{[a,b]} \gamma'(t) f(\gamma(t))\, \mathrm{d}\lambda(t).$$

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The integral along a chain $\Gamma = \gamma_1 \dotplus \cdots \dotplus \gamma_n$ in $\mathbb{C}$ of a continuous mapping $f\colon \langle\Gamma\rangle \to X$ is defined by

$$\int_\Gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

Lemma 44

Let Γ be a chain in $\mathbb{C}$, X a complex Banach space, $f\colon \langle\Gamma\rangle \to X$ continuous, and $\phi \in X^*$. Then $\phi\big(\int_\Gamma f\big) = \int_\Gamma \phi \circ f$.

If $\Omega \subset \mathbb{C}$ is open and $K \subset \Omega$ compact, then we say that a cycle Γ surrounds K in Ω if $\langle \Gamma \rangle \subset \Omega \setminus K$, $\operatorname{ind}_\Gamma z = 1$ for $z \in K$, and $\operatorname{ind}_\Gamma z = 0$ for $z \in \mathbb{C} \setminus \Omega$.

Theorem 45

Let $\Omega \subset \mathbb{C}$ *be open,* X *a complex Banach space, and let* $f\colon \Omega \to X$ *be holomorphic. If* Γ_1, Γ_2 *are two cycles in* Ω *such that* $\mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\Gamma_2}(z)$ *for every* $z \in \mathbb{C} \setminus \Omega$*, then* $\int_{\Gamma_1} f = \int_{\Gamma_2} f$*.*

Theorem 45

Let $\Omega \subset \mathbb{C}$ be open, X a complex Banach space, and let $f\colon \Omega \to X$ be holomorphic. If Γ_1, Γ_2 are two cycles in Ω such that $\mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\Gamma_2}(z)$ *for every $z \in \mathbb{C} \setminus \Omega$, then $\int_{\Gamma_1} f = \int_{\Gamma_2} f$.*

Definition 46

Let A be a complex Banach algebra with a unit and $x \in A$. If $f \in H(\Omega)$, where $\Omega \subset \mathbb{C}$ is an open neighbourhood of $\sigma(x)$, then we define

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} f R_x = \frac{1}{2\pi i} \int_{\Gamma} f(\alpha)(\alpha e - x)^{-1} \, \mathrm{d}\alpha,$$

where Γ is any cycle surrounding $\sigma(x)$ in Ω.

Lemma 47

Let (Ω, μ) be a space with a complete measure, A a Banach algebra and $f \in L_1(\mu, A)$. Then

$$x\left(\int_E f d\mu\right) = \int_E x f(t)\,\mathrm{d}\mu(t) \quad \text{and} \quad \left(\int_E f\,\mathrm{d}\mu\right)x = \int_E f(t) x\,\mathrm{d}\mu(t)$$

for every $x \in A$ and every measurable $E \subset \Omega$.

Fact 48

Let G be a group. If $u, v \in G$ commute, then also u, v, u^{-1}, v^{-1} commute.

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Lemma 49

Let A be an algebra with a unit, $x \in A$, and $\mu, \nu \in \rho(x)$.

(a) $R_x(\mu)R_x(\nu) = R_x(\nu)R_x(\mu)$.

(b) $R_x(\mu) - R_x(\nu) = (\nu - \mu)R_x(\mu)R_x(\nu)$ *(resolvent identity).*

Theorem 50 (holomorphic calculus)

Let A be a complex Banach algebra with a unit, $x \in A$, $\Omega \subset \mathbb{C}$ an open neighbourhood of $\sigma(x)$, and $f \in H(\Omega)$. The mapping $\Phi\colon H(\Omega) \to A$, where $\Phi(g) = g(x)$ from Definition 46, has the following properties:

(a) *Consider $H(\Omega)$ with the topology of locally uniform convergence. Then Φ is a continuous algebra homomorphism for which $\Phi(1) = e$ and $\Phi(Id) = x$.*

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(b) *$f(x) \in A^\times$ if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$. In this case $f(x)^{-1} = \frac{1}{f}(x)$.*

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(c) *$\sigma(f(x)) = f(\sigma(x))$ (spectral mapping theorem).*

(d) *If $g \in H(\Omega_1)$, where Ω_1 is an open neighbourhood of $f(\sigma(x))$, then $(g \circ f)(x) = g(f(x))$.*

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(c) *$\sigma(f(x)) = f(\sigma(x))$ (spectral mapping theorem).*

(d) *If $g \in H(\Omega_1)$, where Ω_1 is an open neighbourhood of $f(\sigma(x))$, then $(g \circ f)(x) = g(f(x))$.*

(e) *If $y \in A$ commutes with x, then y commutes also with $f(x)$.*

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(e) *If $y \in A$ commutes with x, then y commutes also with $f(x)$.*

(f) *If B is a complex Banach algebra and $\Theta\colon A \to B$ a continuous homomorphism such that $\Theta(e)$ is a unit in B, then $f(\Theta(x)) = \Theta(f(x))$. In particular, if $z \in A^\times$, then $f(zxz^{-1}) = zf(x)z^{-1}$.*

L47, L49(b), T45, C41; L23; F48; P28, T43

4. Multiplicative linear functionals

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Definition 51

Let A be an algebra over $\mathbb{K}$. A homomorphism $\varphi\colon A \to \mathbb{K}$ is called a multiplicative linear functional (i.e. φ is linear and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$).

4. Multiplicative linear functionals

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Proposition 52

Let A be an algebra over $\mathbb{K}$. Then $\Delta(A)$ is a linearly independent set.

Proposition 53

Let A be an algebra. Every multiplicative linear functional φ on A has a unique extension $\widetilde{\varphi} \in \Delta(A_e)$ given by $\widetilde{\varphi}(x, \lambda) = \varphi(x) + \lambda$ and $\Delta(A_e) = \{\widetilde{\varphi};\ \varphi \in \Delta(A) \cup \{0\}\}$.

Proposition 54

Let A be an algebra and $\varphi \in \Delta(A)$. Then $\varphi(x) \in \sigma(x)$ for every $x \in A$ and so $|\varphi(x)| \leq r(x)$.

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Corollary 55

Let A be a Banach algebra. Then $\Delta(A) \subset B_{A^*}$ (in particular, every multiplicative linear functional on A is automatically continuous). If A has a unit, then $\|\varphi\| \geq \frac{1}{\|e\|}$ for every $\varphi \in \Delta(A)$. In particular, if $\|e\| = 1$, then $\Delta(A) \subset S_{A^*}$.

Definition 56

Let A be an algebra. A maximal ideal in A is a proper ideal in A that is maximal with respect to the ordering of all proper ideals in A by inclusion.

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Let A be an algebra with a unit. Then every proper ideal in A is contained in some maximal ideal in A.

Proposition 58

Let A be a Banach algebra with a unit. If I is a proper ideal in A, then also $\overline{I}$ is a proper ideal in A. So every maximal ideal in A is closed.

Lemma 59

Let A be a commutative algebra with a unit and suppose that $x \in A$ is not invertible. Then the principal ideal xA is proper.

Theorem 60

Let A be a complex commutative Banach algebra with a unit and let I be a proper ideal in A. Then there exists $\varphi \in \Delta(A)$ such that $\varphi\restriction_I = 0$.

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Corollary 61

If A is a non-trivial complex commutative Banach algebra with a unit, then $\Delta(A) \neq \emptyset$.

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Corollary 62

Let A be a complex commutative Banach algebra with a unit. Then the mapping $\Phi : \varphi \mapsto \mathrm{Ker}\,\varphi$ is a bijection between $\Delta(A)$ and the set of all maximal ideals in A.

Theorem 63

Let A be a Banach algebra and $M = \Delta(A) \cup \{0\} \subset (B_{A^*}, w^*)$ is the set of all linear multiplicative functionals on A. Then M is compact, $\Delta(A)$ is locally compact, and if A has a unit, then $\Delta(A)$ is compact. If $\Delta(A)$ is not compact, then M is the Alexandrov compactification of $\Delta(A)$.

Theorem 63

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The mapping $\Phi\colon M \to \Delta(A_{\mathrm{e}})$, where $\Phi(\varphi) = \widetilde{\varphi}$ is the unique extension of φ to the element of $\Delta(A_{\mathrm{e}})$, is a homeomorphism.

Let X, Y be vector spaces and $T\colon X \to Y$ be a linear mapping. Then we define the algebraically dual mapping $T^{\#}\colon Y^{\#} \to X^{\#}$ by the formula $T^{\#}f(x) = f(Tx)$ for $f \in Y^{\#}$ and $x \in X$.

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Lemma 64

Let X, Y be vector spaces and $T\colon X \to Y$ a linear bijection. Then $T^\#$ is a bijection and $(T^\#)^{-1} = (T^{-1})^\#$.

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Lemma 64

Let X, Y be vector spaces and $T\colon X \to Y$ a linear bijection. Then $T^\#$ is a bijection and $(T^\#)^{-1} = (T^{-1})^\#$.

Proposition 65

Let A, B be Banach algebras and $\Phi\colon A \to B$ an algebraic isomorphism. Then the mapping $\Psi = \Phi^\# \restriction_{\Delta(B)}$ is a homeomorphism of $\Delta(B)$ onto $\Delta(A)$.

Proposition 66

Let S, T be topological spaces and let $h\colon S \to T$ be continuous and onto. Then $\Phi\colon C_{\mathrm{b}}(T) \to C_{\mathrm{b}}(S)$, $\Phi(f) = f \circ h$ is an isometric isomorphism of the Banach algebra $C_{\mathrm{b}}(T)$ into the Banach algebra $C_{\mathrm{b}}(S)$.

Proposition 66

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Theorem 67

Let K, L be locally compact Hausdorff topological spaces. Then the following statements are equivalent:

(i) *The Banach algebras $C_0(K)$ and $C_0(L)$ are isometrically isomorphic.*

(ii) *The algebras $C_0(K)$ and $C_0(L)$ are algebraically isomorphic.*

(iii) *The spaces K and L are homeomorphic.*

Definition 68

A commutative algebra A is called semi-simple if $\Delta(A)$ separates the points of A, i.e. if $\bigcap\{\operatorname{Ker}\varphi;\ \varphi \in \Delta(A)\} = \{0\}$.

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Theorem 69

Let A, B be Banach algebras and suppose B is commutative and semi-simple. Then every homomorphism from A to B is automatically continuous. Also every conjugate-linear multiplicative mapping from A to B is automatically continuous.

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Theorem 69

Let A, B be Banach algebras and suppose B is commutative and semi-simple. Then every homomorphism from A to B is automatically continuous. Also every conjugate-linear multiplicative mapping from A to B is automatically continuous.

Corollary 70

Let A be a commutative semi-simple algebra. Then all norms on A in which A is a Banach algebra are equivalent.

5. Gelfand transform

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Definition 71

Let A be a Banach algebra over $\mathbb{K}$. For $x \in A$ we define $\widehat{x} \colon \Delta(A) \to \mathbb{K}$ by the formula $\widehat{x}(\varphi) = \varphi(x)$, i.e. $\widehat{x} = \varepsilon_x \restriction_{\Delta(A)}$. The function $\widehat{x}$ is called the Gelfand transform of the element x.

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Theorem 72

Let A be a complex commutative Banach algebra and $x \in A$. If A has a unit, then $\operatorname{Rng} \widehat{x} = \sigma(x)$. If A does not have a unit, then $\sigma(x) \setminus \{0\} \subset \operatorname{Rng} \widehat{x} \subset \sigma(x)$.

5. Gelfand transform

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Corollary 73

Let A be a complex commutative Banach algebra and $x \in A$. Then $\|\widehat{x}\|_{C_0(\Delta(A))} = r(x)$.

Definition 74

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Let A be a Banach algebra and let Γ be its Gelfand transform. Then the following hold:

(a) *Γ is a continuous homomorphism and $\|\Gamma\| \leq 1$.*

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Let A be a Banach algebra and let Γ be its Gelfand transform. Then the following hold:

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(a) *Γ is a continuous homomorphism and $\|\Gamma\| \leq 1$.*

(b) *The subalgebra $\Gamma(A) \subset C_0(\Delta(A))$ separates the points of $\Delta(A)$.*

(c) *Γ is one-to-one if and only if $\Delta(A)$ separates the points of A, i.e. if and only if A is commutative and semi-simple.*

Theorem 76

Let A be a complex commutative Banach algebra and let Γ be its Gelfand transform. Then the following hold:

(a) *Γ is an isomorphism into if and only if there exists $K > 0$ such that $\|x^2\| \geq K\|x\|^2$ for every $x \in A$.*

(b) *Γ is an isometry into if and only if $\|x^2\| = \|x\|^2$ for every $x \in A$.*

Definition 77

Let A be a groupoid and $M \subset A$. Then the set $M^{\mathrm{c}} = \{a \in A;\ ax = xa \text{ for every } x \in M\}$, i.e. the set of all elements of A commuting with every element of M, is called the commutant of the set M.

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Proposition 78

Let A be a groupoid and $M \subset A$. Then the following hold:

(a) $M \subset (M^{\mathrm{c}})^{\mathrm{c}}$.

(b) *The set $M \cap M^{\mathrm{c}}$ commutes.*

(c) *If M commutes, then also $(M^{\mathrm{c}})^{\mathrm{c}}$ commutes.*

Proposition 79

Let A be an algebra and $M \subset A$. Then the following hold:

(a) *M^{c} is a subalgebra of A.*

(b) *If A has a unit, then $e \in M^{\mathrm{c}}$.*

(c) *If A is normed, then M^{c} is closed.*

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(c) *If A is normed, then M^{c} is closed.*

Proposition 80

Let A be an algebra with a unit e and suppose that $M \subset A$ commutes. Then $B = (M^{c})^{c}$ is a commutative algebra with a unit e, $M \subset B$, and $B^{\times} = A^{\times} \cap B$. So $\sigma_A(x) = \sigma_B(x)$ for every $x \in B$.

Theorem 81

Let A be a complex Banach algebra and suppose that $x, y \in A$ commute. Then the following hold:

(a) $\sigma(x + y) \subset \sigma(x) + \sigma(y)$ *and* $\sigma(xy) \subset \sigma(x)\sigma(y)$.

(b) $r(x + y) \leq r(x) + r(y)$ *and* $r(xy) \leq r(x)r(y)$.

6. B*-algebras

Theorem 82

Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then there exists a unique operator $T^\star \in \mathcal{L}(H_2, H_1)$ such that

$$\langle Tx, y\rangle_{H_2} = \langle x, T^\star y\rangle_{H_1}$$

for every $y \in H_2$ and $x \in H_1$.

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$$\langle Tx, y\rangle_{H_2} = \langle x, T^\star y\rangle_{H_1}$$

for every $y \in H_2$ and $x \in H_1$. Further, $T^\star = I_1^{-1} \circ T^* \circ I_2$, where $I_j \colon H_j \to H_j^*$, $j = 1, 2$ are the corresponding conjugate-linear isometries from the Löwig-Fréchet-Riesz theorem.

Theorem 82

Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then there exists a unique operator $T^\star \in \mathcal{L}(H_2, H_1)$ such that

$$\langle Tx, y\rangle_{H_2} = \langle x, T^\star y\rangle_{H_1}$$

for every $y \in H_2$ and $x \in H_1$. Further, $T^\star = I_1^{-1} \circ T^* \circ I_2$, where $I_j\colon H_j \to H_j^*$, $j = 1, 2$ are the corresponding conjugate-linear isometries from the Löwig-Fréchet-Riesz theorem.

Definition 83

The operator $T^\star$ from the preceding theorem is called the hilbertian adjoint operator to T.

Theorem 84

Let H_1, H_2, H_3 be Hilbert spaces.

(a) *If $T \in \mathcal{L}(H_1, H_2)$, then $T^{\star\star} = (T^\star)^\star = T$.*

Theorem 84

Let H_1, H_2, H_3 be Hilbert spaces.

(a) *If $T \in \mathcal{L}(H_1, H_2)$, then $T^{\star\star} = (T^\star)^\star = T$.*

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(c) *Let $T \in \mathcal{L}(H_1, H_2)$ and $S \in \mathcal{L}(H_2, H_3)$. Then $(S \circ T)^\star = T^\star \circ S^\star$. Also, $(Id_{H_1})^\star = Id_{H_1}$.*

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(d) *Let $T \in \mathcal{L}(H_1, H_2)$. Then $\|T^\star \circ T\| = \|T \circ T^\star\| = \|T\|^2$.*

(e) *$T^\star$ is an isomorphism if and only if T is an isomorphism.*

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(d) *Let $T \in \mathcal{L}(H_1, H_2)$. Then $\| T^\star \circ T\| = \| T \circ T^\star \| = \| T \|^2$.*

(e) *$T^\star$ is an isomorphism if and only if T is an isomorphism.*

(f) *$T^\star$ is compact if and only if T is compact.*

Definition 85

Let A be an algebra over $\mathbb{K}$. The mapping $^\star : A \to A$ is called an algebra involution if it has the following properties:

- $(x + y)^\star = x^\star + y^\star$ for every $x, y \in A$,
- $(\lambda x)^\star = \overline{\lambda} x^\star$ for every $x \in A$ and $\lambda \in \mathbb{K}$,
- $(xy)^\star = y^\star x^\star$ for every $x, y \in A$,
- $(x^\star)^\star = x$ for every $x \in A$ (i.e. the mapping * is an involution).

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- $(x^\star)^\star = x$ for every $x \in A$ (i.e. the mapping * is an involution).

An algebra on which there is an algebra involution is called an algebra with an involution.

Fact 86

Let A be an algebra with an involution. Then $(a, \alpha)^\star = (a^\star, \overline{\alpha})$ for $(a, \alpha) \in A_\mathrm{e}$ is an algebra involution on A_e that extends the involution from A.

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Proposition 87

Let A be an algebra with an involution and $x \in A$. Then the following hold:

(a) *If e is a left or right unit in A, then e is a unit and $e^\star = e$.*

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(b) *Suppose A has a unit. Then $x^\star \in A^\times$ if and only if $x \in A^\times$. In this case $(x^\star)^{-1} = (x^{-1})^\star$.*

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(b) *Suppose A has a unit. Then $x^\star \in A^\times$ if and only if $x \in A^\times$. In this case $(x^\star)^{-1} = (x^{-1})^\star$.*

(c) *$\lambda \in \sigma(x)$ if and only if $\overline{\lambda} \in \sigma(x^\star)$. Therefore $r(x^\star) = r(x)$.*

Proposition 88

Let A be a commutative semi-simple Banach algebra. Then every algebra involution on A is continuous.

Definition 89

Let A be an algebra with an involution. An element $x \in A$ is called self-adjoint if $x^\star = x$.

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Let A be an algebra with an involution and $x, y \in A$. Then the following hold:

(a) *The elements $x + x^\star$, $x^\star x$, $xx^\star$, and in the complex case also $i(x - x^\star)$ are self-adjoint.*

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(c) *If A is complex, then there exist unique self-adjoint elements $u, v \in A$ such that $x = u + iv$. Then $x^\star = u - iv$.*

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(d) *If x, y are self-adjoint and commute, then xy is self-adjoint.*

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(e) *If x is self-adjoint, then $yxy^\star$ is self-adjoint.*

Definition 91

A Banach algebra A with an involution is called a B⋆-algebra if

$$\|x^\star x\| = \|x\|^2$$

for every $x \in A$.

Definition 91

A Banach algebra A with an involution is called a B*-algebra if

$$\|x^\star x\| = \|x\|^2$$

for every $x \in A$.

Lemma 92

Let A be a normed algebra with an involution. Then the following statements are equivalent:

(i) $\|x^\star x\| \geq \|x\|^2$ *for every* $x \in A$.

(ii) $\|xx^\star\| \geq \|x\|^2$ *for every* $x \in A$.

(iii) $\|x^\star x\| = \|x\|^2$ *for every* $x \in A$.

(iv) $\|xx^\star\| = \|x\|^2$ *for every* $x \in A$.

In all cases then $\|x^\star\| = \|x\|$ *for every* $x \in A$.

Proposition 93

Let A be a $B^\star$-algebra without a unit. Then there exists a norm $|||\cdot|||$ on A_e with the involution from Fact 86 extending the original norm on A (and equivalent to the norm from Proposition 9) such that A_e is a $B^\star$-algebra.

Definition 94

Let A be an algebra with an involution.

- If A has a unit, then an element $x \in A$ is called unitary if $x^\star x = xx^\star = e$, or in other words $x^{-1} = x^\star$.

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Let A be an algebra over $\mathbb{K}$ with an involution and $x, y \in A$.

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Let A be an algebra over $\mathbb{K}$ with an involution and $x, y \in A$.

(a) *If A has a unit and if x, y are unitary, then xy is unitary.*

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(a) *If A has a unit and if x, y are unitary, then xy is unitary.*

(b) *If x is normal, then x^n is normal for every $n \in \mathbb{N}$.*

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(c) *If A has a unit and if x is normal and y is unitary, then $yxy^\star$ is normal.*

(d) *If A has a unit and if x is normal and $\lambda \in \mathbb{K}$, then $\lambda e - x$ is normal.*

Theorem 96

Let A be a $B^\star$-algebra and $x \in A$.

(a) *If x is normal, then $\|x^n\| = \|x\|^n$ for every $n \in \mathbb{N}$ and if A is complex, then $r(x) = \|x\|$.*

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(c) *If A has a unit and x is unitary, then $\sigma(x) \subset \{\lambda \in \mathbb{K}; |\lambda| = 1\}$. If moreover A is non-trivial, then $\|x\| = 1$.*

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(c) *If A has a unit and x is unitary, then $\sigma(x) \subset \{\lambda \in \mathbb{K}; |\lambda| = 1\}$. If moreover A is non-trivial, then $\|x\| = 1$.*

(d) *If x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.*

Corollary 97

Let A be a non-trivial complex commutative $B^\star$-algebra. Then $\Delta(A) \neq \emptyset$.

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Let A be a non-trivial complex commutative $B^\star$-algebra. Then $\Delta(A) \neq \emptyset$.

Corollary 98

Let A be a complex algebra with an involution. Then there exists at most one norm on A with which A is a $B^\star$-algebra.

Definition 99

Let A and B be algebras with an involution. Then an algebra homomorphism $\Phi\colon A \to B$ is called a *-homomorphism if it preserves the operation *, i.e. if $\Phi(x^\star) = \Phi(x)^\star$ for every $x \in A$.

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Corollary 100

Let A be a complex $B^\star$-algebra. Then every multiplicative linear functional on A is a *-homomorphism.

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Corollary 101

Let A, B be complex $B^\star$-algebras and $\Phi\colon A \to B$ a *-homomorphism. Then Φ is automatically continuous and moreover $\|\Phi\| \leq 1$.

Corollary 102

Let A be a complex $B^\star$-algebra and B its $B^\star$-subalgebra. If A and B has a common unit, then $B^\times = A^\times \cap B$.

Corollary 102

Let A be a complex $B^\star$-algebra and B its $B^\star$-subalgebra. If A and B has a common unit, then $B^\times = A^\times \cap B$. Further, let $x \in B$. If B has a unit which is not a unit in A, then $\sigma_A(x) = \sigma_B(x) \cup \{0\}$, in the other cases $\sigma_A(x) = \sigma_B(x)$.

P87(b), F90(a), T96(d), T32(d), F13, P27

Theorem 103 (I. M. Gelfand a M. A. Naĭmark (1943))

Let A be a complex commutative $B^\star$-algebra. Then the Gelfand transform is an isometric $\star$-isomorphism of A onto $C_0(\Delta(A))$.

Corollary 104

A complex commutative $B^\star$-algebra A has a unit if and only if $\Delta(A)$ is compact.

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Corollary 105

Let A and B are complex commutative $B^\star$-algebras. Then the following statements are equivalent:

(i) *A and B are isometrically $\star$-isomorphic.*

(ii) *A and B are algebraically isomorphic.*

(iii) *The spaces $\Delta(A)$ and $\Delta(B)$ are homeomorphic.*

Theorem 106 (I. M. Gelfand a M. A. Naĭmark (1943), I. Kaplansky (1953))

Every complex $B^\star$-algebra can be embedded by an isometric $\star$-isomorphism into $\mathscr{L}(H)$ for some suitable complex Hilbert space H.

7. Continuous calculus for normal elements of B⋆-algebras

7. Continuous calculus for normal elements of B*-algebras

Proposition 107

Let A be a normed algebra over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f, g \colon \Omega \to A$, and $t \in \Omega$. If $f'(t)$ and $g'(t)$ exist, then
$(fg)'(t) = f'(t)g(t) + f(t)g'(t)$.

Let A be a (real) Banach algebra with a unit and $x \in A$. Then we define

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Theorem 108

Let A be a Banach algebra over $\mathbb{K}$ with a unit e and $x \in A$.

(a) *If $y \in A$ commutes with x, then* $\exp x \exp y = \exp(x + y)$.

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(a) *If $y \in A$ commutes with x, then* $\exp x \exp y = \exp(x + y)$.

(b) $\exp x \in A^\times$ *and* $(\exp x)^{-1} = \exp(-x)$.

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Let A be a Banach algebra over $\mathbb{K}$ with a unit e and $x \in A$.

(a) *If $y \in A$ commutes with x, then* $\exp x \exp y = \exp(x + y)$.

(b) $\exp x \in A^{\times}$ *and* $(\exp x)^{-1} = \exp(-x)$.

(c) *Put $f(\lambda) = \exp(\lambda x)$ for $\lambda \in \mathbb{K}$. Then $f'(\lambda) = \exp(\lambda x)x$ for every $\lambda \in \mathbb{K}$.*

Theorem 108

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(c) *Put $f(\lambda) = \exp(\lambda x)$ for $\lambda \in \mathbb{K}$. Then $f'(\lambda) = \exp(\lambda x)x$ for every $\lambda \in \mathbb{K}$.*

(d) *If A is an algebra with a continuous involution, then* $(\exp x)^\star = \exp x^\star$.

Theorem 108

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(a) *If $y \in A$ commutes with x, then* $\exp x \exp y = \exp(x + y)$.

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(c) *Put $f(\lambda) = \exp(\lambda x)$ for $\lambda \in \mathbb{K}$. Then $f'(\lambda) = \exp(\lambda x)x$ for every $\lambda \in \mathbb{K}$.*

(d) *If A is an algebra with a continuous involution, then* $(\exp x)^{\star} = \exp x^{\star}$.

(e) *If A is a complex algebra with a continuous involution and x is self-adjoint, then $\exp(ix)$ is unitary.*

Theorem 109 (Bent Fuglede (1950), Calvin R. Putnam (1951))

Let A be a complex $B^\star$-algebra, $x \in A$, and let $a, b \in A$ be normal and such that $ax = xb$. Then $a^\star x = xb^\star$.

Definition 110

Let A be an algebra and $M \subset A$. The set

$$\operatorname{alg} M = \bigcap \{B \supset M;\ B \text{ is a subalgebra of } A\}$$

is called algebra hull of M.

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Proposition 111

Let A be an algebra and $M \subset A$. Then

$$\operatorname{alg} M = \operatorname{span}\{x_1 x_2 \cdots x_n;\ x_1, \dots, x_n \in M, n \in \mathbb{N}\}.$$

Definition 112

Let A be a normed algebra and $M \subset A$. Then we define a closed algebra hull of M as

$$\overline{\operatorname{alg}}\, M = \bigcap \{B \supset M;\ B \text{ is a closed subalgebra of } A\}.$$

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Proposition 113

Let A be a normed algebra and $M \subset A$. Then $\overline{\operatorname{alg}}\, M = \overline{\operatorname{alg} M}$.

Fact 114

Let A, B be algebras and $M \subset A$. Then every algebra homomorphism $\Phi\colon \operatorname{alg} M \to B$ is uniquely determined by its values on M.

Fact 114

Let A, B be algebras and $M \subset A$. Then every algebra homomorphism $\Phi\colon \operatorname{alg} M \to B$ is uniquely determined by its values on M. If A, B are normed algebras, then every continuous algebra homomorphism $\Phi\colon \overline{\operatorname{alg}} M \to B$ is uniquely determined by its values on M.

Proposition 115

Let A be a $B^\star$-algebra and suppose that $M \subset A$ commutes and is closed under the involution. Then $\overline{\operatorname{alg}} M$ is a commutative $B^\star$-subalgebra of A.

Theorem 116

Let A be an algebra over $\mathbb{K}$ with a unit and $x \in A$. Let $\Omega_2 \subset \mathbb{K}$ be closed and $\Omega_1 \subset \Omega_2$. Let $\Phi_i \colon C(\Omega_i) \to A$ be an algebra homomorphism such that $\Phi_i(1) = e$, $\Phi_i(Id) = x$, in the complex case moreover $\Phi_1(\overline{Id}) = \Phi_2(\overline{Id})$, and let Φ_i be sequentially continuous from the topology of locally uniform convergence on $C(\Omega_i)$ to some Hausdorff topology τ on A, $i = 1, 2$. Then $\Phi_1(f\restriction_{\Omega_1}) = \Phi_2(f)$ for every $f \in C(\Omega_2)$.

Let A be a complex B*-algebra with a unit and let $x \in A$ be normal. Set $B = \overline{\text{alg}}\{e, x, x^\star\}$. Then we can define

$$f(x) = \Gamma_B^{-1}(f \circ \Gamma_B(x)). \tag{1}$$

Theorem 117 (continuous calculus)

Let A be a complex $B^\star$-algebra with a unit, let $x \in A$ be normal and $f \in C(\sigma(x))$. The mapping $\Phi\colon C(\sigma(x)) \to A$, where $\Phi(g) = g(x)$ is given by the formula (1), *has the following properties:*

(a) *Φ is an isometric $\star$-isomorphism of $C(\sigma(x))$ onto $B = \overline{\mathrm{alg}}\{e, x, x^\star\}$, for which moreover $\Phi(1) = e$ and $\Phi(Id) = x$.*

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(c) $f(x) \in A^\times$ *if and only if* $f(\lambda) \neq 0$ *for every* $\lambda \in \sigma(x)$. *In this case* $f(x)^{-1} = \frac{1}{f}(x)$.

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(e) *$\sigma(f(x)) = f(\sigma(x))$ (spectral mapping theorem).*

P66, T103; C102, F14; P22; C101, T116

(f) If $C \subset A$ is a commutative B*-subalgebra containing e and x, then $\Gamma_C^{-1}(f \circ \Gamma_C(x)) = f(x)$.

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(j) If D is a complex B*-algebra and $\Theta\colon A \to D$ is a *-homomorphism such that $\Theta(e)$ is a unit in D, then $f(\Theta(x)) = \Theta(f(x))$. In particular, if $u \in A$ is unitary, then $f(uxu^\star) = uf(x)u^\star$.

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If A does not have a unit, then we carry out the whole construction in A_{e}. If $f \in C(\sigma(x))$ is such that $f(0) = 0$, then $f(x) \in A$.

Theorem 119

Let A be a complex $B^\star$-algebra and $x \in A$.

(a) *The element x is self-adjoint if and only if it is normal and $\sigma(x) \subset \mathbb{R}$.*

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(a) *The element x is self-adjoint if and only if it is normal and $\sigma(x) \subset \mathbb{R}$.*

(b) *If A has a unit, then x is unitary if and only if it is normal and $\sigma(x) \subset \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$.*

8. Non-negative elements of B⋆-algebras

Definition 120

Let A be an algebra with an involution and let $x \in A$ be self-adjoint. We say that x is non-negative, if $\sigma(x) \subset [0, +\infty)$.

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An element x of a complex $B^\star$-algebra is non-negative, if and only if it is normal and $\sigma(x) \subset [0, +\infty)$.

Proposition 122

Let A be an algebra with an involution and let $x, y \in A$ be non-negative.

(a) *If $t \geq 0$, then tx is non-negative.*

(b) *If A is a complex $B^\star$-algebra, then $x + y$ is non-negative.*

(c) *If A is a complex Banach algebra and x and y commute, then xy is non-negative.*

Fact 123

Let A be a complex $B^\star$-algebra and $x \in A$.

(a) *If x is non-negative, then $|x| = x$.*

(b) *If x is self-adjoint, then $|x|^2 = x^2$.*

(c) *If x is non-negative, then $(\sqrt{x}\,)^2 = x$. Moreover, $\sqrt{x}$ is the only non-negative $y \in A$ satisfying $y^2 = x$.*

(d) *If x is self-adjoint, then $\sqrt{x^2} = |x|$.*

Proposition 124

Let A be a complex $B^\star$-algebra. Then for every self-adjoint element $x \in A$ there exists a unique pair of non-negative elements $x^+, x^- \in A$ such that $x = x^+ - x^-$ and $x^- x^+ = x^+ x^- = 0$. Moreover, $x^+ + x^- = |x|$.

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Theorem 125 (I. Kaplansky (1953))

Let A be a complex $B^\star$-algebra and $x \in A$. Then $x^\star x$ and $xx^\star$ are non-negative.

Theorem 126 (polar decomposition)

Let A be a complex $B^\star$-algebra with a unit and let $x \in A$ be invertible. Then there exist a unitary $u \in A$ and a non-negative $a \in A$ satisfying $x = ua$. This decomposition is unique.

II. Continuous linear operators on Hilbert spaces

1. Basic properties

Theorem 127

If H_1, H_2 are Hilbert spaces and $T \in \mathscr{L}(H_1, H_2)$, then

(a) $\text{Ker}\, T^\star = (\text{Rng}\, T)^\perp$,

(b) $\text{Ker}\, T = (\text{Rng}\, T^\star)^\perp$,

(c) $\overline{\text{Rng}\, T} = (\text{Ker}\, T^\star)^\perp$,

(d) $\overline{\text{Rng}\, T^\star} = (\text{Ker}\, T)^\perp$.

Definition 128

Let X, Y, and Z be vector spaces over $\mathbb{K}$. A mapping $B\colon X \times Y \to Z$ is called bilinear if it is linear separately in the first and in the second coordinate, i.e. the mapping $x \mapsto B(x, y)$ is linear for every $y \in Y$ and $y \mapsto B(x, y)$ is linear for every $x \in X$.

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Proposition 129 (polarisation formula)

Let X, Y be vector spaces over $\mathbb{K}$ and let $S\colon X \times X \to Y$ be a sesquilinear mapping. Then

$$S(x,y) + S(y,x) = \frac{1}{2}\big(S(x+y,x+y) - S(x-y,x-y)\big)$$

for every $x, y \in X$.

Proposition 129 (polarisation formula)

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$$S(x,y) + S(y,x) = \frac{1}{2}\big(S(x+y,x+y) - S(x-y,x-y)\big)$$

for every $x, y \in X$. If $\mathbb{K} = \mathbb{C}$, then

$$S(x,y) = \frac{1}{4}\big(S(x+y,x+y) - S(x-y,x-y) + \\ + iS(x+iy,x+iy) - iS(x-iy,x-iy)\big)$$

for every $x, y \in X$.

Theorem 130

Let X be an inner-product space and let $T\colon X \to X$ be a linear operator. Suppose moreover that at least one of the following condition holds:

- *X is complex.*
- *X is a Hilbert space and T is continuous and self-adjoint.*

If $\langle Tx, x\rangle = 0$ for every $x \in X$, then $T = 0$.

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Corollary 131

Let X be an inner-product space and let $S, T\colon X \to X$ be linear operators. Suppose moreover that at least one of the following condition holds:

- *X is complex.*
- *X is a Hilbert space and S, T are continuous and self-adjoint.*

If $\langle Sx, x\rangle = \langle Tx, x\rangle$ for every $x \in X$, then $S = T$.

Definition 132

Let X, Y, Z be normed linear spaces and let $B\colon X \times Y \to Z$ be a bilinear, resp. sesquilinear mapping. We say that B is **bounded** if $\sup_{x\in B_X, y\in B_Y} \|B(x,y)\| < +\infty$. In this case we define $\|B\| = \sup_{x\in B_X, y\in B_Y} \|B(x,y)\|$.

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Proposition 133

Let H be a Hilbert space. If S is a bounded sesquilinear form on H, then there exists a unique $T \in \mathcal{L}(H)$ such that $S(x,y) = \langle Tx, y \rangle$ for all $x, y \in H$. Moreover, $\|T\| = \|S\|$.

Fact 134

Let H_1, H_2 be Hilbert spaces and $T \in \mathscr{L}(H_1, H_2)$. Then $\operatorname{Ker} T^\star \circ T = \operatorname{Ker} T$.

Theorem 135

Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then the following statements are equivalent:

(i) *T is normal.*

(ii) $\langle T^\star x, T^\star y \rangle = \langle Tx, Ty \rangle$ *for every $x, y \in H$.*

(iii) $\|T^\star x\| = \|Tx\|$ *for every $x \in H$.*

Definition 136

Let X be a normed linear spacer over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. A number $\lambda \in \mathbb{K}$ is called an approximate eigenvalue of the operator T if there exists a sequence $\{x_n\} \subset S_X$ such that $(\lambda I - T)x_n \to 0$.

Definition 136

Let X be a normed linear spacer over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. A number $\lambda \in \mathbb{K}$ is called an approximate eigenvalue of the operator T if there exists a sequence $\{x_n\} \subset S_X$ such that $(\lambda I - T)x_n \to 0$. The set of all approximate eigenvalues of the operator T is called an approximate point spectrum of the operator T and it is denoted by $\sigma_{\text{ap}}(T)$.

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Fact 137

Let X be a normed linear space over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. Then $\lambda \in \mathbb{K}$ is an approximate eigenvalue of T if and only if $\lambda I - T$ is not an isomorphism into.

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Proposition 138

Let X, Y be normed linear spaces, $T \in \mathcal{L}(X)$, and let $S\colon X \to Y$ be a linear isomorphism. Then $\sigma_{\mathrm{ap}}(S \circ T \circ S^{-1}) = \sigma_{\mathrm{ap}}(T)$, where $S \circ T \circ S^{-1} \in \mathcal{L}(Y)$.

Definition 139

Let X be an inner-product space and $T \in \mathscr{L}(X)$. The set $N_T = \{\langle Tx, x\rangle;\ x \in S_X\}$ is called a numerical range of the operator T.

Fact 140

Let X be a normed linear space with $\dim X_{\mathbb{R}} \neq 1$ (i.e. X is either complex, or real of dimension not equal to 1). Then S_X is pathwise connected.

Proposition 141

Let X be an inner-product space over $\mathbb{K}$ and $T \in \mathcal{L}(X)$.

(a) $N_{\alpha I + \beta T} = \alpha + \beta N_T$ *for any* $\alpha, \beta \in \mathbb{K}$.

Proposition 141

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(a) $N_{\alpha I+\beta T} = \alpha + \beta N_T$ *for any* $\alpha, \beta \in \mathbb{K}$.

(b) *The set N_T is pathwise connected.*

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(a) $N_{\alpha I+\beta T} = \alpha + \beta N_T$ *for any* $\alpha, \beta \in \mathbb{K}$.

(b) *The set N_T is pathwise connected.*

(c) $\sigma_{\mathrm{p}}(T) \subset N_T \subset B_{\mathbb{K}}(0, \|T\|)$.

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(b) *The set N_T is pathwise connected.*

(c) $\sigma_{\mathrm{p}}(T) \subset N_T \subset B_{\mathbb{K}}(0, \|T\|)$.

(d) $\sigma_{\mathrm{ap}}(T) \subset \overline{N_T}$. *If X is a Hilbert space, then* $\sigma(T) \setminus \sigma_{\mathrm{ap}}(T) \subset N_T$, *and so* $\sigma(T) \subset \overline{N_T}$.

Theorem 142

Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be normal. Then the following hold:

(a) $\operatorname{Ker} T = \operatorname{Ker} T^\star$.

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(a) $\operatorname{Ker} T = \operatorname{Ker} T^{\star}$.

(b) $\operatorname{Rng} T$ *is dense in* H *if and only if* T *is one-to-one.*

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(a) $\operatorname{Ker} T = \operatorname{Ker} T^\star$.

(b) $\operatorname{Rng} T$ *is dense in H if and only if T is one-to-one.*

(c) *T is invertible if and only if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in H$.*

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(d) $\sigma(T) = \sigma_{\text{ap}}(T)$.

T135; T127(c); F137

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(d) $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

(e) *$\lambda \in \sigma_{\mathrm{p}}(T)$ if and only if $\overline{\lambda} \in \sigma_{\mathrm{p}}(T^\star)$. The eigenspace of T corresponding to an eigenvalue λ is equal to the eigenspace of $T^\star$ corresponding to the eigenvalue $\overline{\lambda}$.*

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(a) $\operatorname{Ker} T = \operatorname{Ker} T^\star$.

(b) $\operatorname{Rng} T$ *is dense in H if and only if T is one-to-one.*

(c) *T is invertible if and only if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in H$.*

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(f) *If λ_1, λ_2 are different eigenvalues of T, then* $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$.

Theorem 143

Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then T is self-adjoint if and only if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every $x, y \in H$.

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(a) $\langle Tx, x\rangle \in \mathbb{R}$ *for every* $x \in H$.

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(a) $\langle Tx, x\rangle \in \mathbb{R}$ *for every* $x \in H$.

(b) $N_T \subset \mathbb{R}$. *If H is non-trivial and if we denote* $m_T = \inf N_T$, $M_T = \sup N_T$, *then* $\|T\| = \max\{|m_T|, |M_T|\}$ *and* $\{m_T, M_T\} \subset \sigma(T) \subset [m_T, M_T]$, *and so the number* $\|T\|$ *or* $-\|T\|$ *lies in* $\sigma(T)$.

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(c) $r(T) = \sup\{|\lambda|;\ \lambda \in N_T\} = \|T\|$.

Proposition 144

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is self-adjoint if and only if $N_T \subset \mathbb{R}$.

Proposition 144

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is self-adjoint if and only if $N_T \subset \mathbb{R}$.

Corollary 145

Let H be a Hilbert space and $T \in \mathcal{L}(H)$. If T is self-adjoint, then $\sigma(T) \subset [0, +\infty)$ if and only if $\langle Tx, x\rangle \geq 0$ for every $x \in H$. If H is complex, then T is non-negative (element of the algebra $\mathcal{L}(H)$) if and only if $\langle Tx, x\rangle \geq 0$ for every $x \in H$.

Theorem 146

Let H be a Hilbert space and let $P \in \mathscr{L}(H)$ be a projection. Then the following statements are equivalent:

(i) *P is self-adjoint.*

(ii) *P is normal.*

(iii) *P is orthogonal.*

(iv) *P is non-negative.*

Theorem 146

Let H be a Hilbert space and let $P \in \mathcal{L}(H)$ be a projection. Then the following statements are equivalent:

(i) *P is self-adjoint.*

(ii) *P is normal.*

(iii) *P is orthogonal.*

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Lemma 147

Let H be a Hilbert space, $S, T \in \mathcal{L}(H)$ and assume that S is self-adjoint. Then $\operatorname{Rng} S \perp \operatorname{Rng} T$ if and only if $ST = 0$.

Definition 148

Let H_1, H_2 be Hilbert spaces. An operator $T \in \mathscr{L}(H_1, H_2)$ is called unitary if $T^\star \circ T = I_{H_1}$ and $T \circ T^\star = I_{H_2}$, or in other words $T^{-1} = T^\star$.

Definition 148

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Theorem 149

Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then the following statements are equivalent:

(i) *T is unitary.*

(ii) *T is onto and $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H$.*

(iii) *T is an isometry onto.*

Lemma 150

Let H_1, H_2 *be Hilbert spaces and* $T \in \mathcal{L}(H_1, H_2)$*. Let* Y *be a closed subspace of* H_2 *such that* $\operatorname{Rng} T \subset Y$ *and let* $S \in \mathcal{L}(H_1, Y)$ *be defined as* $Sx = Tx$ *for* $x \in H_1$*. Then* $S^\star = T^\star \restriction_Y$*.*

Theorem 151

Let H be a Hilbert space. Then $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$.

Definition 152

Let A be a set and let $f\colon A \to A$ be a mapping. A set $B \subset A$ is called invariant with respect to f if $f(B) \subset B$, i.e. $f\restriction_B\colon B \to B$.

Fact 153

Let H be a Hilbert space, $T \in \mathcal{L}(H)$, and let $M \subset H$ be a set of eigenvectors of T (not necessarily all).

(a) *If $Y \subset H$ is invariant with respect to T, then $Y^\perp$ is invariant with respect to $T^\star$.*

(b) $\overline{\text{span}}\, M$ *is invariant with respect to T.*

(c) *If T normal, then both* $\overline{\text{span}}\, M$ *and* $(\overline{\text{span}}\, M)^\perp$ *are invariant with respect to both T and $T^\star$.*

(d) *Let $Y \subset H$ be a closed subspace invariant with respect to both T and $T^\star$. Then $(T\restriction_Y)^\star = T^\star\restriction_Y$. So if T is self-adjoint, resp. normal, then $T\restriction_Y \in \mathcal{L}(Y)$ is self-adjoint, resp. normal.*

Theorem 154 (spectral decomposition of a normal compact operator; D. Hilbert (1904), Erhard Schmidt (1907))

Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Suppose further that

- *T is self-adjoint or*
- *H is complex and T is normal.*

Then there exist an orthonormal basis B of H consisting of eigenvectors of T. The set of all vectors from B corresponding to non-zero eigenvalues of T is countable and if we enumerate it by an arbitrary injective sequence $\{e_n\}_{n=1}^{N}$, $N \in \mathbb{N}_0 \cup \{\infty\}$, then $\{e_n\}$ is an orthonormal basis of $\overline{\operatorname{Rng} T}$ and

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n \rangle e_n$$

for every $x \in H$, where λ_n is the eigenvalue corresponding to the eigenvector e_n.

Theorem 154 (spectral decomposition of a normal compact operator; D. Hilbert (1904), Erhard Schmidt (1907))

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$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n \rangle e_n$$

for every $x \in H$, where λ_n is the eigenvalue corresponding to the eigenvector e_n.

If $\{\lambda_n\}_{n=1}^{M}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ is an injective sequence of all eigenvalues of T and P_n is the orthogonal projection onto $\operatorname{Ker}(\lambda_n I - T)$, then

$$I = \sum_{n=1}^{M} P_n,$$

where the series converges pointwise unconditionally (i.e. $x = \sum_{n=1}^{M} P_n x$ unconditionally for every $x \in H$) and

$$T = \sum_{n=1}^{M} \lambda_n P_n,$$

where the series converges unconditionally in the space $\mathcal{L}(H)$.

Theorem 155 (representation of a compact operator; E. Schmidt (1907))

Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{K}(H_1, H_2)$. Then there exist $N \in \mathbb{N}_0 \cup \{\infty\}$, a sequence of positive numbers $\{\lambda_n\}_{n=1}^N$, and orthonormal systems $\{u_n\}_{n=1}^N \subset H_1$ and $\{v_n\}_{n=1}^N \subset H_2$ such that

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n$$

for every $x \in H$.

Theorem 155 (representation of a compact operator; E. Schmidt (1907))

Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{K}(H_1, H_2)$. Then there exist $N \in \mathbb{N}_0 \cup \{\infty\}$, a sequence of positive numbers $\{\lambda_n\}_{n=1}^N$, and orthonormal systems $\{u_n\}_{n=1}^N \subset H_1$ and $\{v_n\}_{n=1}^N \subset H_2$ such that

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n$$

for every $x \in H$. Further, $\{\lambda_n^2\}_{n=1}^N$ is a sequence of all non-zero eigenvalues of the operator $T^\star \circ T$, and for every $\lambda > 0$ the number of elements of the set $\{n \in \mathbb{N};\ \lambda_n^2 = \lambda\}$ is equal to $\dim \operatorname{Ker}(\lambda I - T^\star \circ T)$. So the sequence $\{\lambda_n\}_{n=1}^N$ is determined uniquely up to a permutation and if $N = \infty$, then $\lambda_n \to 0$.

2. Bounded Borel calculus

Definition 156

Let X, Y be normed linear spaces. We define the following locally convex topologies on the space $\mathscr{L}(X, Y)$:

- the strong operator topology τ_{SOT} is generated by the system of seminorms $\{p_x(T) = \|Tx\|;\ x \in X\}$,

Definition 156

Let X, Y be normed linear spaces. We define the following locally convex topologies on the space $\mathcal{L}(X, Y)$:

- the strong operator topology τ_{SOT} is generated by the system of seminorms $\{p_x(T) = \|Tx\|;\ x \in X\}$,
- the weak operator topology τ_{WOT} is generated by the system of seminorms $\{p_{x,f}(T) = |f(Tx)|;\ x \in X, f \in Y^*\}$.

Definition 156

Let X, Y be normed linear spaces. We define the following locally convex topologies on the space $\mathcal{L}(X, Y)$:

- the strong operator topology τ_{SOT} is generated by the system of seminorms $\{p_x(T) = \|Tx\|;\ x \in X\}$,
- the weak operator topology τ_{WOT} is generated by the system of seminorms $\{p_{x,f}(T) = |f(Tx)|;$ $x \in X, f \in Y^*\}$.

The symbol $\mathrm{Bf_b}(X)$ denotes the set of all bounded Borel functions on a topological space X.

Definition 157

Let X be a Banach space over $\mathbb{K}$ and $T \in \mathscr{L}(X)$. We say that a mapping $\Psi\colon \mathrm{Bf_b}(\sigma(T)) \to \mathscr{L}(X)$ is a Borel functional calculus for T if Ψ is an algebra homomorphism, $\Psi(1) = I$, $\Psi(Id) = T$, and if $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to $f \in \mathrm{Bf_b}(\sigma(T))$, then $\Psi(f_n) \to \Psi(f)$ in the topology τ_{WOT}.

Let A be an algebra over $\mathbb{K}$ with a unit, τ a Hausdorff topology on A, $x, y \in A$, and $F \subset \mathbb{K}$ closed. A homomorphism $\Phi \colon \mathrm{Bf_b}(F) \to A$ will be called a Borel calculus on F for τ and a pair (x, y) if $\Phi(1) = e$, $\Phi(Id) = x$, $\Phi(\overline{Id}) = y$, and $\Psi(f_n) \overset{\tau}{\to} \Psi(f)$ whenever $\{f_n\} \subset \mathrm{Bf_b}(F)$ is a bounded sequence converging pointwise to $f \in \mathrm{Bf_b}(F)$.

Let A be an algebra over $\mathbb{K}$ with a unit, τ a Hausdorff topology on A, $x, y \in A$, and $F \subset \mathbb{K}$ closed. A homomorphism $\Phi\colon \mathrm{Bf_b}(F) \to A$ will be called a Borel calculus on F for τ and a pair (x, y) if $\Phi(1) = e$, $\Phi(Id) = x$, $\Phi(\overline{Id}) = y$, and $\Psi(f_n) \xrightarrow{\tau} \Psi(f)$ whenever $\{f_n\} \subset \mathrm{Bf_b}(F)$ is a bounded sequence converging pointwise to $f \in \mathrm{Bf_b}(F)$.

Theorem 158

Let A be a Banach algebra over $\mathbb{K}$ with a unit, τ a Hausdorff topology on A (non-strictly) weaker than norm, and $x, y \in A$. Assume that there exists a Borel calculus Ψ on a closed $F \subset \mathbb{K}$ for τ and a pair (x, y). Then there is a Borel calculus Φ on $\sigma(x)$ for τ and a pair (x, y). If moreover Ψ_1 is a Borel calculus on F_1 for τ and a pair (x, y), then $\Psi_1(f) = \Phi(f\restriction_{\sigma(x)})$ for every $f \in \mathrm{Bf_b}(F_1)$.

Lemma 159

Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty} \subset H$. If $x_n \to x \in H$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$ (in the norm).

Let H be a complex Hilbert space and let $T \in \mathcal{L}(H)$ be a normal operator. For fixed $x, y \in H$ consider the function $\varphi_{x,y} \colon C(\sigma(T)) \to \mathbb{C}$ defined by

$$\varphi_{x,y}(f) = \langle f(T)x, y \rangle.$$

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There exist a regular Borel complex measure $\mu_{x,y}$ on $\sigma(T)$ such that

$$\varphi_{x,y}(f) = \int_{\sigma(T)} f \, \mathrm{d}\mu_{x,y}$$

for every $f \in C(\sigma(T))$, and $\|\mu_{x,y}\| = \|\varphi_{x,y}\| \leq \|x\| \|y\|$.

Let H be a complex Hilbert space and let $T \in \mathcal{L}(H)$ be a normal operator. For fixed $x, y \in H$ consider the function $\varphi_{x,y} \colon C(\sigma(T)) \to \mathbb{C}$ defined by

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for every $f \in C(\sigma(T))$, and $\|\mu_{x,y}\| = \|\varphi_{x,y}\| \leq \|x\| \|y\|$. For $f \in \mathrm{Bf_b}(\sigma(T))$ there exist a unique operator $f(T) \in \mathcal{L}(H)$ such that

$$\langle f(T)x, y \rangle = \int_{\sigma(T)} f \, \mathrm{d}\mu_{x,y} \tag{2}$$

for every $x, y \in H$. Moreover, $\|f(T)\| \leq \|f\|_\infty$.

Theorem 160

Let H be a complex Hilbert space, let $T \in \mathcal{L}(H)$ be a normal operator and $f \in \mathrm{Bf_b}(\sigma(T))$. The mapping $\Phi : \mathrm{Bf_b}(\sigma(T)) \to \mathcal{L}(H)$, where $\Phi(g) = g(T)$ is defined above, is a Borel functional calculus for T with the following properties:

(a) *Φ is a *-homomorphism and if we denote by Ψ the continuous calculus for T from Theorem 117, then $\Phi \restriction_{C(\sigma(T))} = \Psi$. If H is non-trivial, then $\|\Phi\| = 1$.*

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(b) *If $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to f, then $\Phi(f_n) \to \Phi(f)$ in the topology τ_{SOT}.*

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(c) *If Ψ is a Borel functional calculus for T which is moreover a *-homomorphism, then $\Psi(g) = \Phi(g)$ for every $g \in \mathrm{Bf_b}(\sigma(T))$.*

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(d) *$f(T)$ is normal. If f is real, then $f(T)$ is self-adjoint. If $|f| = 1$, then $f(T)$ is unitary.*

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(c) *If Ψ is a Borel functional calculus for T which is moreover a *-homomorphism, then $\Psi(g) = \Phi(g)$ for every $g \in \mathrm{Bf_b}(\sigma(T))$.*

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(f) *If $g \in \mathrm{Bf_b}\big(\overline{\operatorname{Rng} f}\big)$, then $(g \circ f)(T) = g(f(T))$.*

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Let H be a complex Hilbert space, let $T \in \mathcal{L}(H)$ be a normal operator and $f \in \mathrm{Bf_b}(\sigma(T))$. The mapping $\Phi\colon \mathrm{Bf_b}(\sigma(T)) \to \mathcal{L}(H)$, where $\Phi(g) = g(T)$ is defined above, is a Borel functional calculus for T with the following properties:

(a) *Φ is a $\star$-homomorphism and if we denote by Ψ the continuous calculus for T from Theorem 117, then $\Phi\restriction_{C(\sigma(T))} = \Psi$. If H is non-trivial, then $\|\Phi\| = 1$.*

(b) *If $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to f, then $\Phi(f_n) \to \Phi(f)$ in the topology τ_{SOT}.*

(c) *If Ψ is a Borel functional calculus for T which is moreover a $\star$-homomorphism, then $\Psi(g) = \Phi(g)$ for every $g \in \mathrm{Bf_b}(\sigma(T))$.*

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(f) *If $g \in \mathrm{Bf_b}\big(\overline{\mathrm{Rng}\, f}\big)$, then $(g \circ f)(T) = g(f(T))$.*

(g) *If $S \in \mathcal{L}(H)$ commutes with T, then S commutes also with $f(T)$.*

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Let H be a complex Hilbert space, let $T \in \mathcal{L}(H)$ be a normal operator and $f \in \mathrm{Bf_b}(\sigma(T))$. The mapping $\Phi\colon \mathrm{Bf_b}(\sigma(T)) \to \mathcal{L}(H)$, where $\Phi(g) = g(T)$ is defined above, is a Borel functional calculus for T with the following properties:

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(b) *If $\{f_n\} \subset \mathrm{Bf_b}(\sigma(T))$ is a bounded sequence converging pointwise to f, then $\Phi(f_n) \to \Phi(f)$ in the topology τ_{SOT}.*

(c) *If Ψ is a Borel functional calculus for T which is moreover a $\star$-homomorphism, then $\Psi(g) = \Phi(g)$ for every $g \in \mathrm{Bf_b}(\sigma(T))$.*

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(f) *If $g \in \mathrm{Bf_b}\big(\overline{\mathrm{Rng}\, f}\big)$, then $(g \circ f)(T) = g(f(T))$.*

(g) *If $S \in \mathcal{L}(H)$ commutes with T, then S commutes also with $f(T)$.*

(h) *If $U \in \mathcal{L}(H)$ is unitary, then $f(UTU^\star) = Uf(T)U^\star$.*

3. Polar decomposition

Theorem 161 (polar decomposition)

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is normal if and only if there exist a unitary $U \in \mathcal{L}(H)$ and a non-negative $A \in \mathcal{L}(H)$ such that $T = UA = AU$. This decomposition is unique if and only if T is one-to one.

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Corollary 162

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$. Then T is normal if and only if there exists a unitary $U \in \mathcal{L}(H)$ such that $T^\star = UT = TU$.

Theorem 163

Let H_1, H_2 be complex Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then there exists a unique pair of operators $A \in \mathcal{L}(H_1)$ and $U \in \mathcal{L}(\overline{\operatorname{Rng} A}, \overline{\operatorname{Rng} T})$ such that $T = U \circ A$, A is non-negative, and U is unitary. If T is an isomorphism, then A is an automorphism of H_1.

Proposition 164

Let $T \in \mathcal{L}(\mathbb{C}^n)$. Then there exist a unitary $U \in \mathcal{L}(\mathbb{C}^n)$ and a non-negative $A \in \mathcal{L}(\mathbb{C}^n)$ such that $T = UA$.

4. Spectral decomposition of an operator

Definition 165

Let $\mathscr{S}$ be a σ-algebra and X a topological vector space. A mapping $\mu\colon \mathscr{S} \to X$ is called a vector measure if $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for every sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets from $\mathscr{S}$.

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Fact 166

Let X, Y be topological vector spaces, $\mu\colon \mathscr{S} \to X$ a vector measure, and $T\colon X \to Y$ a continuous linear mapping. Then $T \circ \mu$ is also a vector measure.

Proposition 167

Let X, Y be normed linear spaces over $\mathbb{K}$, $\mathcal{S}$ a σ-algebra, and $\mu\colon \mathcal{S} \to (\mathcal{L}(X, Y), \tau_{\mathrm{WOT}})$ a vector measure. Then for every $x \in X$ and $f \in Y^*$ the function $\mu_{x,f}\colon \mathcal{S} \to \mathbb{K}$ given by

$$\mu_{x,f}(A) = f(\mu(A)x)$$

is a complex measure on $\mathcal{S}$. The mapping $B\colon (x, f) \mapsto \mu_{x,f}$ is a bilinear mapping from $X \times Y^*$ to a normed linear space of complex measures on $\mathcal{S}$. If moreover X is a Banach space, then $\sup_{A\in\mathcal{S}} \|\mu(A)\| < +\infty$ and B is bounded.

Theorem 168 (B. J. Pettis (1938))

Let X be a normed linear space and $\mu : \mathscr{S} \to (X, w)$ a vector measure. Then μ is also a vector measure as a mapping into $(X, \|\cdot\|)$.

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Corollary 169

Let X, Y be normed linear spaces, $\mathcal{S}$ a σ-algebra, and $\mu \colon \mathcal{S} \to (\mathcal{L}(X, Y), \tau_{\mathrm{WOT}})$ a vector measure. Then μ is also a vector measure as a mapping into $(\mathcal{L}(X, Y), \tau_{\mathrm{SOT}})$.

By $\mathrm{Bs}(X)$ we denote the σ-algebra of Borel subsets of a topological space X.

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Definition 170

Let X be a Banach space over $\mathbb{K}$. A resolution of the identity on X is a vector measure $E\colon \mathrm{Bs}(\mathbb{K}) \to (\mathcal{L}(X), \tau_{\mathrm{SOT}})$ with the following properties:

(i) $E(A)$ is a projection for every Borel $A \subset \mathbb{K}$.

(ii) $E(\mathbb{K}) = I$.

(iii) $E(A \cap B) = E(A)E(B)$ for every Borel $A, B \subset \mathbb{K}$.

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(ii) $E(\mathbb{K}) = I$.

(iii) $E(A \cap B) = E(A)E(B)$ for every Borel $A, B \subset \mathbb{K}$.

If X is a Hilbert space and all projections $E(A)$ are orthogonal, then E is called an orthogonal resolution of the identity on X.

Fact 171

Let X be a Banach space over $\mathbb{K}$ and E a resolution of the identity on X.

(a) *The projections $E(A)$ and $E(B)$ commute for every $A, B \in \mathrm{Bs}(\mathbb{K})$.*

Fact 171

Let X be a Banach space over $\mathbb{K}$ and E a resolution of the identity on X.

(a) *The projections $E(A)$ and $E(B)$ commute for every $A, B \in \mathrm{Bs}(\mathbb{K})$.*

(b) *If $A, B \in \mathrm{Bs}(\mathbb{K})$, $B \subset A$, then* $\operatorname{Rng} E(B) \subset \operatorname{Rng} E(A)$ *and* $\operatorname{Ker} E(B) \supset \operatorname{Ker} E(A)$.

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(e) *If $A, B \in \mathrm{Bs}(\mathbb{K})$ are disjoint, then* $\operatorname{Rng} E(A) \perp \operatorname{Rng} E(B)$.

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(f) *$E_{x,x}$ is a finite regular Borel non-negative measure on $\mathbb{K}$ and $\|E_{x,x}\| = \|x\|^2$ for every $x \in X$.*

Lemma 172

Let X be a Banach space over $\mathbb{K}$ and suppose $E\colon \mathrm{Bs}(\mathbb{K}) \to \mathcal{L}(X)$ has the following properties:

(i) *$E(A)$ is a projection for every Borel $A \subset \mathbb{K}$.*

(ii) *$E(\mathbb{K}) = I$.*

(iii) *$E(A \cap B) = E(A)E(B)$ for every Borel $A, B \subset \mathbb{K}$.*

(iv) *$E_{x,f}\colon \mathrm{Bs}(\mathbb{K}) \to \mathbb{K}$, $E_{x,f}(A) = f(E(A)x)$ is a Borel complex measure on $\mathbb{K}$ for every $x \in X$ and $f \in X^*$.*

Then E is a resolution of the identity on X.

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Then E is a resolution of the identity on X.
If X is a complex Hilbert space, then instead of (iv) it suffices to assume that $E_{x,x}\colon \mathrm{Bs}(\mathbb{K}) \to \mathbb{C}$, $E_{x,x}(A) = \langle E(A)x, x\rangle$ is a finite Borel measure on $\mathbb{C}$ for every $x \in X$.

Proposition 173

Let X, Y be Banach spaces over $\mathbb{K}$, let E be a resolution of the identity on X, and let $S\colon X \to Y$ be a linear isomorphism. Then $F\colon A \mapsto S \circ E(A) \circ S^{-1}$, $A \in \mathrm{Bs}(\mathbb{K})$ is a resolution of the identity on Y.

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Definition 174

Let X be a Banach space over $\mathbb{K}$ and $T \in \mathcal{L}(X)$. We say that E is a resolution of the identity with respect to the operator T if E is a resolution of the identity on X such that for every Borel $A \subset \mathbb{K}$ the following holds:

(i) the projection $E(A)$ commutes with T,

(ii) if we set $T_A = T\restriction_{\operatorname{Rng} E(A)}$, then $\sigma(T_A) \subset \bar{A}$.

Proposition 175

Let X be a Banach space over $\mathbb{K}$, $T \in \mathcal{L}(X)$, and E a resolution of the identity with respect to T.

(a) $\sigma(T_A) \subset \sigma(T)$ *for every Borel* $A \subset \mathbb{K}$.

Proposition 175

Let X be a Banach space over $\mathbb{K}$, $T \in \mathcal{L}(X)$, and E a resolution of the identity with respect to T.

(a) $\sigma(T_A) \subset \sigma(T)$ *for every Borel* $A \subset \mathbb{K}$*.*

(b) *In the complex case* $E(\sigma(T)) = I$*.*

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(c) *If* $E(\sigma(T)) = I$ *(in particular if X is complex), then* $E(G) \neq 0$ *for every (relatively) open non-empty* $G \subset \sigma(T)$.

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(d) $\operatorname{Ker}(\lambda I - T) \subset \operatorname{Rng} E(\{\lambda\})$ *for every* $\lambda \in \mathbb{K}$. *In particular, if λ is an eigenvalue of T, then* $E(\{\lambda\}) \neq 0$.

Lemma 176

Let X, Y be normed linear spaces, $T \in \mathcal{L}(X)$, let $Z \subset X$ be a subspace invariant with respect to T, and let $S\colon X \to Y$ be a linear isomorphism. Then $S(Z)$ is invariant with respect to $U = S \circ T \circ S^{-1} \in \mathcal{L}(Y)$ and $\sigma(U\restriction_{S(Z)}) = \sigma(T\restriction_{Z})$.

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Proposition 177

Let X, Y be Banach spaces over $\mathbb{K}$, $T \in \mathcal{L}(X)$, and $S\colon X \to Y$ a linear isomorphism. If E is a resolution of the identity with respect to T, then $F\colon A \mapsto S \circ E(A) \circ S^{-1}$, $A \in \mathrm{Bs}(\mathbb{K})$, is a resolution of the identity with respect to the operator $U = S \circ T \circ S^{-1} \in \mathcal{L}(Y)$.

Theorem 178

Let X be a Banach space over $\mathbb{K}$. If Ψ is a Borel functional calculus for $T \in \mathcal{L}(X)$, then there exists a resolution of the identity E with respect to T such that

$$\phi(Tx) = \int_{\sigma(T)} \lambda \, \mathrm{d}E_{x,\phi}(\lambda)$$

for every $x \in X$ and $\phi \in X^*$. This resolution has the following properties:

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$$\phi(\Psi(f)x) = \int_{\sigma(T)} f \, \mathrm{d}E_{x,\phi}$$

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(e) *If X is complex and λ an isolated point of $\sigma(T)$, then $\lambda \in \sigma_{\mathrm{p}}(T)$.*

(f) *If X is a Hilbert space and Ψ is a $\star$-homomorphism, then E is orthogonal.*

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for every $f \in \mathrm{Bf_b}(\sigma(T))$ and every $x \in X$ and $\phi \in X^*$.

(c) *$E(\{\lambda\})$ is a projection onto $\mathrm{Ker}(\lambda I - T)$ for every $\lambda \in \mathbb{K}$.*

(d) *$\lambda \in \sigma_{\mathrm{p}}(T)$ if and only if $E(\{\lambda\}) \neq 0$.*

(e) *If X is complex and λ an isolated point of $\sigma(T)$, then $\lambda \in \sigma_{\mathrm{p}}(T)$.*

(f) *If X is a Hilbert space and Ψ is a *-homomorphism, then E is orthogonal.*

On the other hand, if E is a resolution of the identity on X such that $E(K) = I$ for some compact $K \subset \mathbb{K}$, then there exists a unique mapping $\Psi \colon \mathrm{Bf_b}(K) \to \mathcal{L}(X)$ such that (b) holds. This Ψ is a Borel functional calculus for $T = \Psi(Id)$, E is a resolution of the identity with respect to T, and (a)–(e) holds. If moreover X is a complex Hilbert space and E is orthogonal, then Ψ is a *-homomorphism and T is normal.

Corollary 179

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$ a normal operator. Then there exists a unique orthogonal resolution of the identity E on H such that there is a compact $K \subset \mathbb{C}$ containing $\sigma(T)$, $E(K) = I$, and

$$\langle Tx, x \rangle = \int_K \lambda \, \mathrm{d}E_{x,x}(\lambda)$$

for every $x \in H$. This resolution is given by the formula $E(A) = \chi_A(T)$. It is an orthogonal resolution of the identity with respect to T.

$$\langle f(T)x, y \rangle = \int_{\sigma(T)} f \, \mathrm{d}E_{x,y}$$

for every $f \in \mathrm{Bf_b}(\sigma(T))$ and every $x, y \in H$. Further, (c), (d), (e) of Theorem 178 hold.

Definition 180

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be measurable spaces, X a topological vector space, $\mu \colon \mathcal{S} \to X$ a vector measure, and $f \colon S \to T$ a measurable mapping. The mapping $f(\mu) \colon \mathcal{T} \to X$ defined by the formula $f(\mu)(A) = \mu(f^{-1}(A))$ for $A \in \mathcal{T}$ is called an image of the vector measure μ.

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Proposition 181

Let X be a Banach space over $\mathbb{K}$, E a resolution of the identity with respect to $T \in \mathscr{L}(X)$ such that $E(K) = I$ for some compact $K \subset \mathbb{K}$, and $f \in \mathrm{Bf_b}(K)$. Then $f(E)$ is a resolution of the identity with respect to $f(T) = \Psi(f)$, where Ψ is the Borel functional calculus for T from Theorem 178.