I. Topological vector spaces

1. Elementary properties

Definition 1. Let X be a vector space over $\mathbb{K}$ and τ a topology on X. If the operations of addition and scalar multiplication are continuous as mappings $+\colon X \times X \to X$ and $\cdot\colon \mathbb{K} \times X \to X$, then the pair (X, τ) is called a *topological vector space*.

The system of all neighbourhoods of a point $x \in X$ is denoted by $\tau(x)$.

Fact 2. *Let X be a vector space and ρ a translation-invariant pseudometric na X. Then*

(a) the operation of addition is continuous as a mapping $+\colon (X, \rho) \times (X, \rho) \to (X, \rho)$ (it is even 2-Lipschitz);

(b) $\rho(nx, 0) \le n\rho(x, 0)$ for every $x \in X$ and $n \in \mathbb{N}$.

Proposition 3. *Let X be a topological vector space over $\mathbb{K}$.*

(a) If $a \in X$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then the operations $x \mapsto x + a$ and $x \mapsto \lambda x$ are homeomorphisms of X onto X.

(b) $\tau(x) = x + \tau(0)$ for every $x \in X$.

(c) If $U \in \tau(0)$, then there exists an open $V \in \tau(0)$ such that $V + V \subset U$.

Definition 4. Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is called

- *absorbing*, if for each $x \in X$ there exists a $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$;

- *balanced*, if $\alpha A \subset A$ for every $\alpha \in \mathbb{K}$, $|\alpha| \le 1$.

Proposition 5. *Let X be a topological vector space.*

(a) Every $U \in \tau(0)$ is absorbing.

(b) $\tau(0)$ has a basis consisting of open balanced sets.

Theorem 6 (John von Neumann (1935)). *Let X be a vector space and $\mathcal{U}$ a system of subsets of X containing 0, which is a basis of a filter (i.e. it is non-empty and for each $U_1, U_2 \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2$). Assume that $\mathcal{U}$ has the following properties:*

(i) For each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V + V \subset U$.

(ii) Each set in $\mathcal{U}$ is absorbing.

(iii) Each set in $\mathcal{U}$ is balanced.

Then there is a unique topology τ on X such that (X, τ) is a topological vector space and $\mathcal{U}$ is a basis of neighbourhoods of 0.

Theorem 7. *Let X be a topological vector space.*

(a) Let $K \subset X$ be compact and $C \subset X$ closed and disjoint from K. Then there exists an open balanced $V \in \tau(0)$ such that $(K + V) \cap (C + V) = \emptyset$.

(b) X is regular (i.e. a point and a closed set can be separated by open sets).

(c) The following statements are equivalent:

(i) X is Hausdorff.
(ii) X is T_1 (i.e. points are closed sets).
(iii) $\{0\}$ is a closed set.
(iv) $\{0\} = \bigcap\{U;\ U \in \tau(0)\}$.

Proposition 8. *Let X be a topological vector space.*

(a) If $G \subset X$ is open and $A \subset X$ arbitrary, then $A + G$ is open.

(b) If $F \subset X$ is closed and $K \subset X$ compact, then $F + K$ is closed.

(c) If $K, L \subset X$ are compact, then $K + L$ is also compact.

Proposition 9. *Let X be a topological vector space over $\mathbb{K}$ and $A, B \subset X$. Then the following hold:*

(a) $\overline{A} = \bigcap\{A + U;\ U \in \tau(0)\}$.

(b) $\overline{A} + \overline{B} \subset \overline{A + B}$ *and* $\operatorname{Int} A + \operatorname{Int} B \subset \operatorname{Int}(A + B)$.

(c) $\lambda \overline{A} = \overline{\lambda A}$ *and* $\lambda \operatorname{Int} A = \operatorname{Int}(\lambda A)$ *for any* $\lambda \in \mathbb{K} \setminus \{0\}$.

(d) *If* Y *is a subspace of* X, *then* $\overline{Y}$ *is also a subspace of* X.

(e) *If* A *is convex, then* $\overline{A}$ *and* $\operatorname{Int} A$ *are convex. Moreover, if* $\operatorname{Int} A$ *is non-empty, then* $\overline{A} = \overline{\operatorname{Int} A}$.

(f) *If* A *is balanced, then* $\overline{A}$ *is balanced.*

(g) *If* A *is balanced and* $0 \in \operatorname{Int} A$, *then* $\operatorname{Int} A$ *is balanced.*

Theorem 10. *Let* X *be a topological vector space,* $Y \subset X$ *a closed subspace and* $Z \subset X$ *a finite-dimensional subspace. Then* $Y + Z$ *is closed.*

Corollary 11. *Let* X *be a Hausdorff topological vector space. Every finite-dimensional subspace of* X *is closed in* X.

2. Bounded sets, metrisability

Definition 12. Let X be a topological vector space and $A \subset X$. The set A is called *bounded* if for every $U \in \tau(0)$ there exists a $t > 0$ such that $A \subset tU$.

Proposition 13. *Let* X *be a topological vector space over* $\mathbb{K}$ *and* $A \subset X$. *Then the following statements are equivalent:*

(i) *The set* A *is bounded.*

(ii) $\gamma_n x_n \to 0$ *for every sequence* $\{x_n\} \subset A$ *and every sequence* $\{\gamma_n\} \subset \mathbb{K}$, $\gamma_n \to 0$.

(iii) $\frac{1}{n} x_n \to 0$ *for every sequence* $\{x_n\} \subset A$.

Proposition 14. *Let* X *be a topological vector space and let* $A, B \subset X$ *be bounded. Then the following hold:*

(a) *The sets* $A \cup B$ *and* $A + B$ *are bounded.*

(b) *The set* λA *is bounded for any* $\lambda \in \mathbb{K}$.

(c) *The set* $\overline{A}$ *is bounded.*

Lemma 15. *Let* X *be a topological vector space over* $\mathbb{K}$, $V \in \tau(0)$ *and* $\{\delta_n\} \subset \mathbb{K} \setminus \{0\}$, $\delta_n \to 0$. *Then* $\{\delta_n V; \ n \in \mathbb{N}\}$ *is a basis of neighbourhoods of* 0 *if and only if* V *is bounded.*

Theorem 16. *Let* X *be a topological vector space. Then the following statements are equivalent:*

(i) X *has a countable basis of neighbourhoods of* 0.

(ii) X *is pseudometrisable.*

(iii) X *is pseudometrisable by a translation-invariant pseudometric.*

If X *is Hausdorff, that the prefix pseudo- above can be omitted.*

Definition 17. Let X be a topological vector space. We say that X is *locally bounded* if every $x \in X$ has a bounded neighbourhood.

Theorem 18. *Let* X *be a locally bounded topological vector space. Then* X *is pseudometrisable.*

3. Total boundedness and compactness

Definition 19. Let X be a topological vector space and $A \subset X$. The set A is called *totally bounded* if for every $U \in \tau(0)$ there exists a finite $F \subset A$ such that $A \subset F + U$.

Proposition 20. *Let* X *be a topological vector space and* $A, B \subset X$. *Then the following hold:*

(a) A *is totally bounded if and only if for every* $U \in \tau(0)$ *there exists a finite* $F \subset X$ *such that* $A \subset F + U$.

(b) *If* A *is totally bounded and* $B \subset A$, *then* B *is also totally bounded.*

(c) *If* A, B *are totally bounded, then also* $A \cup B$ *and* $A + B$ *are totally bounded.*

(d) *If* A *is totally bounded and* $\lambda \in \mathbb{K}$, *then also* λA *is totally bounded.*

(e) *If* A *is totally bounded, then also* $\overline{A}$ *is totally bounded.*

Proposition 21. *Let X be a topological vector space. Compact subsets of X are totally bounded and totally bounded subsets of X are bounded.*

Fact 22. *Let (X, τ) be a topological vector space pseudometrisable by a translation-invariant pseudometric ρ. Then $A \subset X$ is τ-totally bounded if and only if it is ρ-totally bounded.*

Definition 23. Let (X, τ_X), (Y, τ_Y) be topological vector spaces and $f : X \to Y$. We say that f is *uniformly continuous* if for every $V \in \tau_Y(0)$ there exists a $U \in \tau_X(0)$ such that $f(x) \in f(y) + V$ whenever $x, y \in X$ are such that $x \in y + U$.

Proposition 24. *Let (X, τ_X), (Y, τ_Y) be topological vector spaces and let $f : X \to Y$ be uniformly continuous. If $A \subset X$ is totally bounded, then $f(A)$ is also totally bounded.*

4. Linear mappings

A linear image of a balanced set is again a balanced set, a pre-image of a balanced set under a linear mapping is again a balanced set.

Theorem 25. *Let X and Y be topological vector spaces and $T : X \to Y$ a linear mapping. Consider the following statements:*

  (i) T is bounded on some neighbourhood of 0.

  (ii) T is continuous at 0.

  (iii) T is continuous.

  (iv) T is uniformly continuous.

  (v) T is sequentially continuous.

  (vi) $T(A)$ is bounded for every bounded $A \subset X$.

  (vii) The set $\{T(x_n); \ n \in \mathbb{N}\}$ is bounded whenever $\{x_n\} \subset X$, $x_n \to 0$.

Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi)$\Leftrightarrow$(vii). If Y is locally bounded, then (i)–(iv) are equivalent. If X is pseudometrisable, then (ii)–(vii) are equivalent.

Lemma 26. *Let X be a pseudometrisable topological vector space. If $\{x_n\} \subset X$ converges to 0, then there exists a sequence $\{\gamma_n\} \subset \mathbb{N}$ such that $\gamma_n \to +\infty$ and $\gamma_n x_n \to 0$.*

Theorem 27. *Let X be a topological vector space over $\mathbb{K}$ and let $f : X \to \mathbb{K}$ be a non-zero linear form. Then the following statements are equivalent:*

  (i) f is continuous.

  (ii) $\operatorname{Ker} f$ is closed.

  (iii) $\overline{\operatorname{Ker} f} \neq X$.

As usual, linear forms will be also called (linear) *functionals*.

Definition 28. Let X be a topological vector space. The symbol $X^{\#}$ denotes the space of all linear forms (functionals) on X and it is called the *algebraic dual*. The symbol X^{*} denotes the subspace of $X^{\#}$ consisting of linear functionals that are continuous on X and it it is called the *topological dual* (or just the *dual*).

Definition 29. Let X and Y be topological vector spaces and $T : X \to Y$ a linear mapping. We say that T is an isomorphism of X onto Y (or just an *isomorphism*) if T is a homeomorphism of X onto Y; we say that T is an isomorphism of X into Y (or just an *isomorphism into*) if T is an isomorphism of X onto $\operatorname{Rng} T$.

Fact 30. *Let X be a vector space, Y a topological vector space, and $\{T_\gamma\}_{\gamma \in \Gamma}$ a net of linear mappings from X into Y. If $T : X \to Y$ is a pointwise limit of the net $\{T_\gamma\}$, then T is linear.*

5. Finite-dimensional spaces

Theorem 31. *Let X be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:*

  (i) X is Hausdorff and $\dim X < \infty$.

  (ii) The exists an $n \in \mathbb{N}$ such that X is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$.

  (iii) X is Hausdorff and has a totally bounded neighbourhood of 0.

  (iv) X is pseudometrisable and every linear mapping from X into some topological vector space is continuous.

  (v) X is pseudometrisable and every linear form on X is continuous.

Corollary 32. *Let X be a finite-dimensional vector space. Then there exists only one Hausdorff vector topology on X.*

6. Locally convex spaces

Fact 33. *Let X be a vector space over $\mathbb{K}$, $A, B \subset X$ convex, and $\alpha \in \mathbb{K}$. Then the sets αA and $A + B$ are convex.*

Definition 34. Let X be a vector space over $\mathbb{K}$. A set $A \subset X$ is called *absolutely convex* if it is convex and balanced.

Fact 35. *Let X be a vector space over $\mathbb{K}$ and $A \subset X$. Then the following hold:*

(a) If A is balanced, then $\operatorname{conv} A$ is balanced and so absolutely convex.

(b) A is absolutely convex if and only if $\alpha x + \beta y \in A$ for every $x, y \in A$ and $\alpha, \beta \in \mathbb{K}$, $|\alpha| + |\beta| \leq 1$.

Fact 36. *Let X be a vector space and p a seminorm on X. Then the following hold:*

(a) $|p(x) - p(y)| \leq p(x - y)$ for every $x, y \in X$.

(b) The set $Z = p^{-1}(0)$ is a subspace of X. If $x, y \in X$ are such that $x - y \in Z$, then $p(x) = p(y)$.

(c) The sets $\{x \in X;\ p(x) < c\}$ and $\{x \in X;\ p(x) \leq c\}$ are absolutely convex for every $c \in [0, +\infty)$.

Definition 37. Let X be a vector space and $f : X \to \mathbb{R}$. We say that f is *positively homogeneous* if $f(tx) = tf(x)$ for every $t \geq 0$.

Fact 38. *Let X be a vector space and f a positively homogeneous function on X. Denote $F_c = \{x \in X;\ f(x) \leq c\}$ and $G_c = \{x \in X;\ f(x) < c\}$ for $c \in \mathbb{R}$. For every $c > 0$ the sets F_c and G_c are absorbing and moreover $F_c = cF_1$, $G_c = cG_1$.*

Definition 39. Let X be a vector space and let $A \subset X$ be absorbing. The *Minkowski functional* of the set A is a function $\mu_A \colon X \to [0, +\infty)$ defined by
$$\mu_A(x) = \inf\{\lambda > 0;\ x \in \lambda A\}.$$

Theorem 40. *Let X be a vector space and let $A \subset X$ be absorbing. Then the following hold:*

(a) If $B \supset A$, then $\mu_B \leq \mu_A$.

(b) μ_A is positively homogeneous.

(c) If A is convex, then μ_A is a non-negative sub-linear functional.

(d) If A is absolutely convex, then μ_A is a seminorm.

(e) $A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

(f) If A is balanced or convex, then $\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\}$.

(g) Let $p \colon X \to [0, +\infty)$ be positively homogeneous and $B \subset X$.

  - *If B is absorbing and $B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B \geq p$.*
  - *If $\{x \in X;\ p(x) < 1\} \subset B$, then $\mu_B \leq p$.*

  So, if $\{x \in X;\ p(x) < 1\} \subset B \subset \{x \in X;\ p(x) \leq 1\}$, then $\mu_B = p$.

Proposition 41. *Let X be a topological vector space and let $A \subset X$ be absorbing. Then $\operatorname{Int} A \subset \{x \in X;\ \mu_A(x) < 1\}$. If moreover A is balanced or convex, then*

$$\operatorname{Int} A \subset \{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\} \subset \overline{A}.$$

Lemma 42. *Let X be a topological vector space and p a sub-linear functional on X. Then p is uniformly continuous if and only if it is bounded above on some neighbourhood of 0.*

Corollary 43. *Let X be a topological vector space and $A \subset X$ an absorbing convex set. Then μ_A is continuous if and only if A is a neighbourhood of 0. In this case*

$$\operatorname{Int} A = \{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leq 1\} = \overline{A}.$$

Theorem 44. *Let X be a topological vector space. Then $X^* \neq \{0\}$ if and only if there is a convex neighbourhood of 0 in X that is different from X.*

Definition 45.

  - We say that a topological vector space is *locally convex* if it has a basis of neighbourhoods of 0 consisting of convex sets.

- A locally convex space whose topology is induced by a complete translation-invariant metric is called a *Fréchet space*.

- We say that a topological vector space is *normable* if its topology is generated by a norm.

Proposition 46. *Let X be a topological vector space. If $U \in \tau(0)$ is convex, then there exists an open absolutely convex $V \in \tau(0)$ such that $V \subset U$.*

Corollary 47. *In a locally convex space $\tau(0)$ has a basis consisting of open absolutely convex absorbing sets and also a basis consisting of closed absolutely convex absorbing sets.*

Let X be a vector space, $p_1, \ldots, p_n$ seminorms on X, and $\varepsilon > 0$. We denote

$$U_{p_1,\ldots,p_n,\varepsilon} = \{x \in X;\ p_1(x) < \varepsilon, \ldots, p_n(x) < \varepsilon\}.$$

Theorem 48. *Let X be a vector space and $\mathcal{P}$ a system of seminorms on X. Then there is a locally convex topology τ on X such that the system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ is a sub-basis of neighbourhoods of 0 and the system $\mathcal{U} = \{U_{p_1,\ldots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0\}$ is a basis of neighbourhoods of 0. The topology τ has the following properties:*

(a) Every seminorm $p \in \mathcal{P}$ is τ-continuous.

(b) A set $A \subset X$ is τ-bounded if and only if $p(A)$ is bounded for each $p \in \mathcal{P}$.

(c) A net $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ converges to $x \in X$ in τ if and only if $p(x_\gamma - x) \to 0$ for each $p \in \mathcal{P}$.

The topology τ will be called a topology generated by the system of seminorms $\mathcal{P}$.

*   On the other hand, if (X, τ) is a locally convex space and $\mathcal{V}$ is a sub-basis of neighbourhoods of 0 consisting of absolutely convex sets, then τ is generated by the system of seminorms $\{\mu_V;\ V \in \mathcal{V}\}$.*

Proposition 49. *Let (X, τ) be a locally convex space. The following statements are equivalent:*

(i) X is Hausdorff.

(ii) Every system of seminorms $\mathcal{P}$ generating τ has the following property:
    For each $x \in X \setminus \{0\}$ there is a $p \in \mathcal{P}$ such that $p(x) > 0$.

(iii) There exists a system of seminorms $\mathcal{P}$ generating τ with the property from statement (ii).

Lemma 50. *Let (X, τ) be a locally convex space generated by a countable system of seminorms $\{p_n\}$. Then*

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{p_n(x - y), 1\}$$

is a translation-invariant pseudometric on X generating τ.

Theorem 51 (A. N. Kolmogorov (1934)). *Let (X, τ) be a topological vector space. Then X is seminormable (resp. normable, if X is Hausdorff) if and only if it has a bounded convex neighbourhood of 0.*

Proposition 52. *Let X be a locally convex space and $A \subset X$. Then the following hold:*

(a) If A is bounded, then also the set conv A is bounded.

(b) If A is totally bounded, then also the set conv A is totally bounded.

7. Separation theorems

Lemma 53. *Let X be a topological vector space and $f \in X^* \setminus \{0\}$. Then f is an open mapping.*

Theorem 54. *Let X be a topological vector space and let $A, B \subset X$ be disjoint convex sets. Then the following hold:*

(a) If A is open, then there exist $f \in X^*$ such that $\operatorname{Re} f(x) < \inf_B \operatorname{Re} f$ for every $x \in A$.

(b) If X is locally convex, A closed and B compact, then there exists $f \in X^*$ such that $\sup_A \operatorname{Re} f < \inf_B \operatorname{Re} f$. If A is moreover absolutely convex, then even $\sup_A |f| < \inf_B \operatorname{Re} f$.

Corollary 55. *Let X be a locally convex space. Then the following hold:*

(a) If X is Hausdorff, then X^* separates the points of X.

(b) If Y is a closed subspace of X and $x \in X \setminus Y$, then there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x) = 1$.

(c) If Y is a subspace of X and $f \in Y^*$, then there exists $F \in X^*$ such that $F\restriction_Y = f$.

8. Weak topologies and polars

Weak topologies

Lemma 56. *Let X be a vector space and let $f, f_1, \ldots, f_n$ be linear forms on X. Then $f \in \operatorname{span}\{f_1, \ldots, f_n\}$ if and only if $\bigcap_{j=1}^{n} \operatorname{Ker} f_j \subset \operatorname{Ker} f$.*

Fact 57. *Let X, Y, Z be vector spaces and $L\colon X \to Y$ and $S\colon X \to Z$ linear mappings. Then there exists a linear mapping $T\colon Z \to Y$ such that $L = T \circ S$ if and only if $\operatorname{Ker} S \subset \operatorname{Ker} L$.*

Definition 58. *Let X be a vector space and let $M \subset X^{\#}$ be non-empty. The symbol $\sigma(X, M)$ denotes the locally convex topology on X generated by the system of seminorms $\{|f|;\ f \in M\}$.*

Proposition 59. *Let X be a vector space and let $M, N \subset X^{\#}$ be non-empty. Then $\sigma(X, M) = \sigma(X, N)$ if and only if $\operatorname{span} M = \operatorname{span} N$. In particular, $\sigma(X, M) = \sigma(X, \operatorname{span} M)$.*

Proposition 60. *Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then the topology $\sigma(X, M)$ is Hausdorff if and only if M separates the points of X.*

Theorem 61. *Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $(X, \sigma(X, M))^* = \operatorname{span} M$.*

Definition 62. *Let X be a topological vector space.*

- *The topology $w = \sigma(X, X^*)$ is called the weak topology on X.*

- *The topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the weak star topology on X^*.*

Corollary 63. *Let (X, τ) be a topological vector space. Then the following hold:*

(a) $w \subset \tau$ and $(X, w)^* = X^*$.

(b) $(X^*, w^*)^* = \varepsilon(X)$.

Proposition 64. *Let X be a Banach space. Then X is reflexive if and only if on the space $(X^*, \|\cdot\|)$ the topologies weak and w^* coincide.*

Proposition 65. *Let X be a topological vector space and Y a subspace of X. Denote by w_{XY} the restriction of the topology $\sigma(X, X^*)$ onto Y. Then $w_{XY} \subset \sigma(Y, Y^*)$. If X is locally convex, then $w_{XY} = \sigma(Y, Y^*)$. In other words, in a locally convex space X the original weak topology on Y coincides with the topology inherited from X.*

Theorem 66. *Let X be a locally convex space and let $A \subset X$ be convex. Then the following hold:*

(a) $\overline{A}^{w} = \overline{A}$.

(b) A is weakly closed if and only if it is closed.

(c) If X is pseudometrisable and $x_n \to x$ weakly, then there exist $y_n \in \operatorname{conv}\{x_j;\ j \geq n\}$ such that $y_n \to x$.

Theorem 67 (George Whitelaw Mackey (1946))**.** *Let X be a locally convex space and $A \subset X$. Then A is bounded if and only if it is weakly bounded.*

Proposition 68. *Let X be a Banach space and $A \subset X^*$. Then A is bounded if and only if it is w^*-bounded.*

Theorem 69. *Let X, Y be topological vector spaces and let $T\colon X \to Y$ be a continuous linear mapping. Then T is w–w continuous, i.e. it is continuous as a mapping $T\colon (X, w) \to (Y, w)$.*

Proposition 70. *Let X be a vector space and let $M \subset X^{\#}$ be non-empty. Then $\sigma(X, M)$ is pseudometrisable if and only if $\operatorname{span} M$ has a countable algebraic basis.*

Proposition 71. *Let X be an infinite-dimensional topological vector space metrisable by a complete metric. Then X does not have a countable algebraic basis.*

Corollary 72.

(a) Let X be a normed linear space. Then (X, w) is metrisable if and only if X is finite-dimensional. In this case the weak topology coincides with the norm topology.

(b) Let X be a Fréchet space. Then (X^*, w^*) is metrisable if and only if X is finite-dimensional.

Polars

Definition 73. Let X be a vector space and $A \subset X$. The *absolutely convex hull* of the set A is defined by

$$\mathrm{aconv}\, A = \bigcap \{B \supset A;\ B \subset X \text{ is absolutely convex}\}.$$

Proposition 74. *Let X be a vector space over $\mathbb{K}$ and $A \subset X$. Then*

$$\mathrm{aconv}\, A = \left\{ \sum_{i=1}^{n} \lambda_i x_i;\ x_1, \dots, x_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{K}, \sum_{i=1}^{n} |\lambda_i| \le 1, n \in \mathbb{N} \right\}.$$

Definition 75. Let X be a topological vector space and $A \subset X$. The *closed absolutely convex hull* of A is defined by

$$\overline{\mathrm{aconv}}\, A = \bigcap \{B \supset A;\ B \subset X \text{ is closed absolutely convex}\}.$$

Proposition 76. *Let X be a topological vector space and $A \subset X$. Then $\overline{\mathrm{span}}\, A = \overline{\mathrm{span}\, A}$, $\overline{\mathrm{conv}}\, A = \overline{\mathrm{conv}\, A}$, and $\overline{\mathrm{aconv}}\, A = \overline{\mathrm{aconv}\, A}$.*

Definition 77. If X is a topological vector space and $A \subset X$, then we define the *(absolute) polar* of the set A by

$$A^{\circ} = \{f \in X^*;\ |f(x)| \le 1 \text{ for every } x \in A\}.$$

For the set $B \subset X^*$ we define the *(absolute) prepolar* by

$$B_{\circ} = \{x \in X;\ |f(x)| \le 1 \text{ for every } f \in B\}.$$

Fact 78. *Let X be a topological vector space, $A \subset X$, and $B \subset X^*$. If we consider X^* with the topology w^*, then $A^{\circ} = \varepsilon(A)_{\circ}$, $\varepsilon(B_{\circ}) = B^{\circ}$, and $(B^{\circ})_{\circ} = (B_{\circ})^{\circ}$.*

Proposition 79. *Let X be a topological vector space over $\mathbb{K}$, $A \subset X$, and $B \subset X^*$. Then the following hold:*

(a) The set A° is absolutely convex and w^*-closed. The set $B_{\circ}$ is absolutely convex and weakly closed.

(b) If A is a subspace of X, then $A^{\circ} = A^{\perp}$. If B is a subspace of X^*, then $B_{\circ} = B_{\perp}$.

(c) $\{0\}^{\circ} = X^*$, $X^{\circ} = \{0\}$, $\{0\}_{\circ} = X$, and if X^* separates the points of X, then $(X^*)_{\circ} = \{0\}$.

(d) If $\lambda \in \mathbb{K} \setminus \{0\}$, then $(\lambda A)^{\circ} = \frac{1}{\lambda} A^{\circ}$ and $(\lambda B)_{\circ} = \frac{1}{\lambda} B_{\circ}$.

(e) If $A_{\gamma} \subset X$, $\gamma \in \Gamma$ is an arbitrary system, then $\left(\bigcup_{\gamma \in \Gamma} A_{\gamma} \right)^{\circ} = \bigcap_{\gamma \in \Gamma} A_{\gamma}^{\circ}$. If $B_{\gamma} \subset X^*$, $\gamma \in \Gamma$ is an arbitrary system, then $\left(\bigcup_{\gamma \in \Gamma} B_{\gamma} \right)_{\circ} = \bigcap_{\gamma \in \Gamma} (B_{\gamma})_{\circ}$.

Theorem 80 (Bipolar theorem; Jean Dieudonné (1950)). *Let X be a topological vector space.*

(a) If $A \subset X$, then $(A^{\circ})_{\circ} = \overline{\mathrm{aconv}}^{\, w} A$ ($= \overline{\mathrm{aconv}}\, A$ if X is locally convex).

(b) If $B \subset X^*$, then $(B_{\circ})^{\circ} = \overline{\mathrm{aconv}}^{\, w^*} B$.

Lemma 81. *Let X be a topological vector space and $A \subset X$, $B \subset X^*$. Then*

(a) $A^{\perp}$ is a w^*-closed subspace of X^*,

(b) $B_{\perp}$ is a weakly closed subspace of X,

(c) $(A^{\perp})_{\perp} = \overline{\mathrm{span}}^{\, w} A$ ($= \overline{\mathrm{span}}\, A$ if X is locally convex),

(d) $(B_{\perp})^{\perp} = \overline{\mathrm{span}}^{\, w^*} B$.

Theorem 82. *If X, Y are topological vector spaces such that Y^* separates the points of Y, and $T : X \to Y$ is a continuous linear mapping, then*

(a) $\mathrm{Ker}\, T^* = (\mathrm{Rng}\, T)^{\perp}$,

(b) $\mathrm{Ker}\, T = (\mathrm{Rng}\, T^*)_{\perp}$,

(c) $\overline{\mathrm{Rng}\, T}^{\, w} = (\mathrm{Ker}\, T^*)_{\perp}$,

(d) $\overline{\mathrm{Rng}\, T^*}^{\, w^*} = (\mathrm{Ker}\, T)^{\perp}$.

Theorem 83 (Herman Heine Goldstine (1938)). *If X is a normed linear space, then $\overline{\varepsilon(B_X)}^{w^*} = B_{X^{**}}$.*

Theorem 84 (Banach-Alaoglu-Bourbaki). *Let X be a topological vector space. If U is a neighbourhood of 0 in X, then U° is a w^*-compact set.*

Corollary 85. *Let X be a normed linear space. Then B_{X^*} is w^*-compact.*

Proposition 86. *Let X be a separable topological vector space and let $\{x_n\}_{n=1}^\infty$ be dense in X. If $U \subset X$ is a neighbourhood of 0, then (U°, w^*) is a topological space metrisable by the metric*

$$\rho(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \min\{|(f - g)(x_n)|, 1\}.$$

Fact 87. *Let X be a normed linear space. Then the canonical embedding $\varepsilon\colon X \to X^{**}$ is an isomorphism of locally convex spaces (X, w) and $(\varepsilon(X), w^*)$. In particular, ε is a homeomorphism of topological spaces (B_X, w) and $(\varepsilon(B_X), w^*)$.*

Proposition 88. *Let X be a normed linear space.*

(a) If X is separable and $\{x_n\}$ is dense in S_X, then (B_{X^*}, w^*) is metrisable by the metric

$$\rho(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} |(f - g)(x_n)|.$$

(b) If X^* is separable and $\{f_n\}$ is dense in S_{X^*}, then (B_X, w) is metrisable by the metric

$$\rho(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} |f_n(x - y)|.$$

Theorem 89. *If X is a Banach space, then X is reflexive if and only if B_X is weakly compact.*

Theorem 90. *Let X be a reflexive Banach space. Then B_X is weakly sequentially compact. That is every bounded sequence in X has a weakly convergent subsequence.*

Proposition 91. *Let X be a normed linear space. Then the mapping $x \mapsto \varepsilon_x \restriction_{B_{X^*}}$ is a linear isometry from X into $C((B_{X^*}, w^*))$. So every normed linear space is isometric to a subspace of $C(K)$ for some Hausdorff compact K.*

II. Theory of distributions

Lemma 92. *Let $\Omega \subset \mathbb{R}^d$ be open.*

(a) Let μ be a Borel complex measure on Ω. If $\int_\Omega \varphi \, d\mu = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $\mu = 0$.

(b) Let $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega f\varphi \, d\lambda = 0$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f = 0$ a. e. on Ω.

(c) Let μ be a Borel complex measure on Ω and $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega \varphi \, d\mu = \int_\Omega f\varphi \, d\lambda$ for every non-negative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f \in L_1(\Omega, \lambda)$ and $\mu(A) = \int_A f \, d\lambda$ for every Borel $A \subset \Omega$.

Lemma 93. *Let $A, U \subset \mathbb{R}^d$ be such that $\mathrm{dist}(A, \mathbb{R}^d \setminus U) > 0$. Then there exists $\varphi \in C^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, $\mathrm{supp}\,\varphi \subset U$, and $\varphi = 1$ on A.*

Corollary 94. *Let $K \subset \mathbb{R}^d$ be compact and let $G \subset \mathbb{R}^d$ be open such that $G \supset K$. Then there exist $U \subset G$ open, $U \supset K$ and $\varphi \in \mathcal{D}(G)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on U.*

1. Weak derivatives

Proposition 95. *Let $(a, b) \subset \mathbb{R}$ and $f \in C^1((a, b))$. Then*

$$\int_a^b f'\varphi \, d\lambda = -\int_a^b f\varphi' \, d\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$.

Definition 96. Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$. We say that a *function* $g \in L_1^{\mathrm{loc}}((a, b))$ is the *weak derivative* of f if

$$\int_a^b g\varphi \, \mathrm{d}\lambda = - \int_a^b f\varphi' \, \mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$. We say that a Borel complex *measure* μ on (a, b) is the *weak derivative* of f if

$$\int_a^b \varphi \, \mathrm{d}\mu = - \int_a^b f\varphi' \, \mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}((a, b))$.

Theorem 97. *The weak derivative of a function $f \in L_1^{\mathrm{loc}}((a, b))$ is uniquely determined. More precisely, if $g_1, g_2 \in L_1^{\mathrm{loc}}((a, b))$ are weak derivatives of f, then $g_1 = g_2$ almost everywhere. If Borel complex measures μ_1, μ_2 on (a, b) are weak derivatives of f, then $\mu_1 = \mu_2$. If $g \in L_1^{\mathrm{loc}}((a, b))$ and a Borel complex measure μ on (a, b) are weak derivatives of f, then $g \in L_1((a, b))$ and $\mu(A) = \int_A g \, \mathrm{d}\lambda$ for every Borel $A \subset (a, b)$.*

Proposition 98. *Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$. Then the weak derivative of f is zero if and only if f is a. e. constant (i.e. there exists $c \in \mathbb{K}$ such that $f = c$ a. e. on (a, b)).*

Theorem 99. *Let $(a, b) \subset \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}((a, b))$.*

(a) *Let $a, b \in \mathbb{R}$. If f is absolutely continuous on $[a, b]$, then it has a finite derivative a. e., $f' \in L_1((a, b))$, and f' is the weak derivative of f. On the other hand, if f has a weak derivative $g \in L_1((a, b))$, then there exists a function f_0 absolutely continuous on $[a, b]$ such that $f = f_0$ a. e. Then $g = f_0'$ a. e.*

(b) *The function f has a weak derivative in $L_1^{\mathrm{loc}}((a, b))$ if and only if there exists a function f_0 locally absolutely continuous on (a, b) such that $f = f_0$ a. e.*

(c) *Let $a, b \in \mathbb{R}$. The function f has a weak derivative equal to a Borel complex measure μ on $[a, b]$ if and only if there exists a left continuous function f_0 of bounded variation on $[a, b]$ such that $f = f_0$ a. e. In this case $\mu([a, x)) = f_0(x) - f_0(a)$ for every $x \in [a, b]$.*

2. The space of test functions and distributions

Definition 100. For $N \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we define

$$\|\varphi\|_N = \max_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty.$$

The sequence of norms $\{\|\cdot\|_N\}_{N=0}^\infty$ on $\mathcal{D}(\mathbb{R}^d)$ generates a Hausdorff locally convex topology τ_ρ metrisable by a translation invariant metric

$$\rho(\varphi, \psi) = \sum_{N=0}^\infty \frac{1}{2^N} \min\{\|\varphi - \psi\|_N, 1\}$$

for $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$.

Theorem 101. *The metric ρ has the following properties:*

(a) *Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The following statements are equivalent:*

   (i) *$\varphi_n \to \varphi$ in the metric ρ.*

   (ii) *$\|\varphi_n - \varphi\|_N \to 0$ for each $N \in \mathbb{N}_0$.*

   (iii) *$D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^d$ for each multi-index α of length d.*

(b) *If α is a multi-index of length d, then the mapping $\varphi \mapsto D^\alpha \varphi$ is continuous as a mapping from $(\mathcal{D}(\mathbb{R}^d), \rho)$ to $(\mathcal{D}(\mathbb{R}^d), \rho)$.*

(c) *$(\mathcal{D}(K), \rho)$ is a complete metric space for every compact $K \subset \mathbb{R}^d$.*

Theorem 102. *Let $\Omega \subset \mathbb{R}^d$ be open. Set*

$$\mathcal{U} = \big\{U \subset \mathcal{D}(\Omega); \ U \text{ absolutely convex}, \ U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\big\}.$$

Then $\mathcal{U}$ is a basis of neighbourhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$ which has the following properties:

(a) *$\tau_\rho \!\restriction_{\mathcal{D}(\Omega)} \subset \tau$.*

(b) $\mathcal{D}(K)$ is a closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau \restriction_{\mathcal{D}(K)} = \tau_K$ for every compact $K \subset \Omega$.

(c) If $A \subset (\mathcal{D}(\Omega), \tau)$ is bounded, then there exists a compact $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$.

(d) Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi_n \to \varphi$ in τ if and only if there exists a compact $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for each $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^d$ for each multi-index α of length d.

(e) If Ω is non-empty, then $(\mathcal{D}(\Omega), \tau)$ is of the first category in itself.

Proposition 103. *Let $\Omega \subset \mathbb{R}^d$ be open, let Y be a locally convex space, and let $T : (\mathcal{D}(\Omega), \tau) \to Y$ be linear. The following statements are equivalent:*

(i) *T is continuous.*

(ii) *The set $\{T(\varphi_n); \ n \in \mathbb{N}\}$ is bounded for every sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ converging to 0 in τ.*

(iii) *For every compact $K \subset \Omega$ the restriction $T \restriction_{\mathcal{D}(K)}$ is continuous.*

Definition 104. Let $\Omega \subset \mathbb{R}^d$ be open. Continuous linear functionals on $(\mathcal{D}(\Omega), \tau)$ are called *distributions* on Ω. The space of all distributions on Ω is therefore the space $\mathcal{D}(\Omega)^* = (\mathcal{D}(\Omega), \tau)^*$.

Theorem 105. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\Lambda : \mathcal{D}(\Omega) \to \mathbb{K}$ be linear. Then $\Lambda \in \mathcal{D}(\Omega)^*$ if and only if for every compact $K \subset \Omega$ there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|\Lambda(\varphi)| \leq C \|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$.*

Definition 106. Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. If there exists $N \in \mathbb{N}_0$ such that for every compact $K \subset \Omega$ there exists $C \geq 0$ such that $|\Lambda(\varphi)| \leq C \|\varphi\|_N$ for every $\varphi \in \mathcal{D}(K)$, then the smallest such N is called the *order* of the distribution Λ. If no such N exists, then the order of Λ is defined as infinity.

3. Operations with distributions

Lemma 107. *Let $k \in \mathbb{N}$, suppose that $f \in C^k(\mathbb{R}^d)$ has all partial derivatives up to order k bounded and let $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$. Then*

$$\int_{\mathbb{R}^d} D^\alpha f \varphi \, d\lambda = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f D^\alpha \varphi \, d\lambda$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Definition 108. Let $\Omega \subset \mathbb{R}^d$ be open and $\Lambda \in \mathcal{D}(\Omega)^*$. For a multi-index α of length d we define the *derivative D^α of the distribution* Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(D^\alpha \Lambda)(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi).$$

For a function $f \in C^\infty(\Omega)$ we define the *product of the function f and the distribution* Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(f\Lambda)(\varphi) = \Lambda(f\varphi).$$

Proposition 109. *Let $\Omega \subset \mathbb{R}^d$ be open, $\Lambda \in \mathcal{D}(\Omega)^*$, $\alpha \in \mathbb{N}_0^d$, and $f \in C^\infty(\Omega)$. Then the following hold:*

(a) $D^\alpha \Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

(c) *If $g \in L_1^{\mathrm{loc}}(\Omega)$, then $f\Lambda_g = \Lambda_{fg}$.*

(d) *If $g \in C^{|\alpha|}(\Omega)$, then $D^\alpha \Lambda_g = \Lambda_{D^\alpha g}$.*

(e) *If $d = 1$, $\Omega = (a, b)$, and $g \in L_1^{\mathrm{loc}}((a, b))$, then*

- $\Lambda_g' = \Lambda_h$, where $h \in L_1^{\mathrm{loc}}((a, b))$, if and only if h is the weak derivative of g;
- $\Lambda_g' = \Lambda_\mu$, where μ is a Borel complex measure on (a, b), if and only if μ is the weak derivative of g.

Fact 110. *Let $\alpha \in \mathbb{N}_0^d$. Then there exist constants $c_\beta^\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}_0^d$, $\beta \leq \alpha$ (the inequality of vectors is understood coordinate-wise) such that for every open $\Omega \subset \mathbb{R}^d$ and every $f, g \in C^{|\alpha|}(\Omega)$ the following holds:*

$$D^\alpha(fg) = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} c_\beta^\alpha D^\beta f D^{\alpha - \beta} g.$$

4. The space of distributions

Proposition 111. *Let $\Omega \subset \mathbb{R}^d$ be open.*

(a) *Let $\alpha \in \mathbb{N}_0^d$ and $g \in C^\infty(\Omega)$. Then the mappings $\Lambda \mapsto D^\alpha \Lambda$ and $\Lambda \mapsto g\Lambda$ are continuous linear mappings of the space $(\mathcal{D}(\Omega)^*, w^*)$ into itself.*

(b) *If $f_n, f \in L_1^{\mathrm{loc}}(\Omega)$ and if $\int_K |f_n - f|\, \mathrm{d}\lambda \to 0$ for each compact $K \subset \Omega$, then $\Lambda_{f_n} \to \Lambda_f$.*

(c) *The mapping $\varphi \mapsto \Lambda_\varphi$ is a one-to-one continuous linear mapping of $(\mathcal{D}(\Omega), \rho)$ into $(\mathcal{D}(\Omega)^*, w^*)$.*

(d) *If $1 \le p \le \infty$ and $f_n \to f$ in $L_p(\Omega)$, then $\Lambda_{f_n} \to \Lambda_f$.*

Theorem 112. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\{\Lambda_n\}$ be a sequence in $\mathcal{D}(\Omega)^*$ such that $\Lambda(\varphi) = \lim_{n \to \infty} \Lambda_n(\varphi)$ exists for every $\varphi \in \mathcal{D}(\Omega)$. Then $\Lambda \in \mathcal{D}(\Omega)^*$.*

5. The support of a distribution

Definition 113. Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. We say that an open set $G \subset \Omega$ is *null* for Λ if $\Lambda(\varphi) = 0$ for every $\varphi \in \mathcal{D}(G)$.

Theorem 114. *Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. The set $G = \bigcup\{H \subset \Omega;\ H \text{ is null for } \Lambda\}$ is null for Λ and it is the largest null set for Λ, i.e. if $H \subset \Omega$ is null for Λ, then $H \subset G$.*

Definition 115. Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω. *The support of the distribution Λ* is defined as $\operatorname{supp} \Lambda = \Omega \setminus G$, where G is the largest null set for Λ.

Theorem 116. *Let $\Omega \subset \mathbb{R}^d$ be open and let Λ be a distribution on Ω.*

(a) *If $f \in C(\Omega)$, then $\operatorname{supp} \Lambda_f = \operatorname{supp} f$.*

(b) *If μ is a Borel complex measure on Ω, then $\operatorname{supp} \Lambda_\mu = \operatorname{supp} \mu$.*

(c) *If $\operatorname{supp} \Lambda$ is compact, then there exist $N \in \mathbb{N}_0$ and $C \ge 0$ such that $|\Lambda(\varphi)| \le C \|\varphi\|_N$ for every $\varphi \in \mathcal{D}(\Omega)$. In particular, Λ is of a finite order.*

(d) *$\operatorname{supp} \Lambda = \{z\}$ for $z \in \Omega$ if and only if $\Lambda = \sum_{|\alpha| \le N} c_\alpha D^\alpha \Lambda_{\delta_z}$ for some $N \in \mathbb{N}_0$ and constants c_α, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le N$ not all zero.*

6. Schwartz space

Lemma 117. *For $N \in \mathbb{N}$ the function $x \mapsto (1 + \|x\|^2)^N$ is a polynomial on $\mathbb{R}^d$. For every polynomial P on $\mathbb{R}^d$ there exist $N \in \mathbb{N}$ and $C > 0$ such that $|P(x)| \le C(1 + \|x\|^2)^N$ for each $x \in \mathbb{R}^d$.*

Definition 118. The *Schwartz space* on $\mathbb{R}^d$ is defined as follows:

$$\mathcal{S}_d = \left\{ f \in C^\infty(\mathbb{R}^d, \mathbb{C});\ PD^\alpha f \text{ is bounded for each } \alpha \in \mathbb{N}_0^d \text{ and each polynomial } P \text{ on } \mathbb{R}^d \right\}.$$

Lemma 119. *Let $d \in \mathbb{N}$, $1 \le p < \infty$, $N > \frac{d}{2p}$, and $h(x) = \frac{1}{(1 + \|x\|^2)^N}$ for $x \in \mathbb{R}^d$. Then $h \in L_p(\mathbb{R}^d)$.*

Proposition 120. *The Schwartz space has the following properties:*

(a) *$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}_d \subset C_0(\mathbb{R}^d) \cap \bigcap_{1 \le p < \infty} L_p(\mathbb{R}^d)$.*

(b) *If $f \in \mathcal{S}_d$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, and $h(x) = f(ax + b)$, then $h \in \mathcal{S}_d$.*

(c) *If $f \in \mathcal{S}_d$ and α is a multi-index of length d, then $D^\alpha f \in \mathcal{S}_d$.*

(d) *If $f \in \mathcal{S}_d$ and if $g \in C^\infty(\mathbb{R}^d)$ is bounded and has all partial derivatives of all orders bounded (in particular, if $g \in \mathcal{S}_d$), then $fg \in \mathcal{S}_d$.*

(e) *If $f \in \mathcal{S}_d$ and $P \colon \mathbb{R}^d \to \mathbb{C}$ is a polynomial, then $Pf \in \mathcal{S}_d$.*

For $N \in \mathbb{N}_0$ and $f \in \mathcal{S}_d$ put

$$\nu_N(f) = \max_{|\alpha| \le N} \left\| x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \right\|_\infty.$$

Denote by σ the Hausdorff locally convex topology on $\mathcal{S}_d$ generated by the system $\{\nu_N\}_{N=0}^\infty$. This topology is metrisable by the metric from Lemma 50.

Theorem 121. *The metric from Lemma 50 corresponding to the system $\{v_N\}_{N=0}^{\infty}$ is complete. The space $(\mathscr{S}_d, \sigma)$ is then a Fréchet space. The topology σ has the following properties:*

(a) *Let $\{f_n\}$ be a sequence in $\mathscr{S}_d$ and $f \in \mathscr{S}_d$. The following statements are equivalent:*

  (i) *$f_n \to f$ in the topology σ.*

  (ii) *$(1 + \|x\|^2)^N D^\alpha f_n \to (1 + \|x\|^2)^N D^\alpha f$ uniformly on $\mathbb{R}^d$ for every $N \in \mathbb{N}_0$ and every multi-index α of length d.*

  (iii) *$PD^\alpha f_n \to PD^\alpha f$ uniformly on $\mathbb{R}^d$ for every polynomial P and every multi-index α of length d.*

(b) *If $f_n \to f$ in the space $(\mathscr{S}_d, \sigma)$, then $f_n \to f$ in $L_p(\mathbb{R}^d)$ for each $1 \le p < \infty$.*

(c) *If α is a multi-index of length d, P is a polynomial on $\mathbb{R}^d$, and $g \in \mathscr{S}_d$, then the mappings $f \mapsto D^\alpha f$, $f \mapsto Pf$, and $f \mapsto gf$ are continuous linear mappings from $(\mathscr{S}_d, \sigma)$ to $(\mathscr{S}_d, \sigma)$.*

Proposition 122. *Let $f \in \mathscr{S}_d$ and $\alpha \in \mathbb{N}_0^d$.*

(a) *$\widehat{D^\alpha f}(t) = (it)^\alpha \widehat{f}(t)$ for every $t \in \mathbb{R}^d$.*

(b) *$D^\alpha \widehat{f} = \widehat{m_\alpha f}$, where $m_\alpha(x) = (-ix)^\alpha$.*

Theorem 123. *The Fourier transform is an isomorphism of the space $(\mathscr{S}_d, \sigma)$ onto itself. Moreover, if $f \in \mathscr{S}_d$, then*

$$\widehat{\widehat{f}}(x) = f(-x) \text{ for every } x \in \mathbb{R}^d \quad \text{and} \quad \widehat{\widehat{\widehat{\widehat{f}}}} = f.$$

7. Tempered distributions

Lemma 124. *Let $K \subset \mathbb{R}^d$ be compact. Then $\sigma \restriction_{\mathscr{D}(K)} = \tau_K$.*

Proposition 125. *The subspace $\mathscr{D}(\mathbb{R}^d)$ is dense in $(\mathscr{S}_d, \sigma)$ and $\sigma \restriction_{\mathscr{D}(\mathbb{R}^d)} \subset \tau$. In other words, the embedding $\mathrm{Id} : (\mathscr{D}(\mathbb{R}^d), \tau) \to (\mathscr{S}_d, \sigma)$ is continuous and onto a dense subset.*

Definition 126. Distributions on $\mathbb{R}^d$ which are restrictions of functionals from $(\mathscr{S}_d, \sigma)^*$ are called *tempered distributions*.

Theorem 127. *Let $\Lambda \in \mathscr{D}(\mathbb{R}^d)^*$. The following statements are equivalent:*

  (i) *Λ is tempered.*

  (ii) *Λ is continuous also in the (weaker) topology σ.*

  (iii) *There exist $N \in \mathbb{N}_0$ and $C \ge 0$ such that $|\Lambda(\varphi)| \le C v_N(\varphi)$ for every $\varphi \in \mathscr{D}(\mathbb{R}^d)$.*

Proposition 128. *Let Λ be a tempered distribution on $\mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $g \in \mathscr{S}_d$, and let P be a polynomial on $\mathbb{R}^d$. Then $D^\alpha \Lambda$, $g\Lambda$, and $P\Lambda$ are also tempered distributions and the formulas*

  - *$D^\alpha \Lambda(f) = (-1)^{|\alpha|} \Lambda(D^\alpha f)$,*

  - *$(g\Lambda)(f) = \Lambda(gf)$, and*

  - *$(P\Lambda)(f) = \Lambda(Pf)$*

hold for every $f \in \mathscr{S}_d$. Further, the mappings $\Lambda \mapsto D^\alpha \Lambda$, $\Lambda \mapsto g\Lambda$, and $\Lambda \mapsto P\Lambda$ are continuous linear mappings from the space $(\mathscr{S}_d^*, w^*)$ into itself.

Definition 129. *The Fourier transform of a tempered distribution Λ on $\mathbb{R}^d$ is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\widehat{f})$ for $f \in \mathscr{S}_d$.*

Theorem 130.

(a) *If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\widehat{g}}$ is a tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\widehat{g}}$. If $g \in L_2(\mathbb{R}^d)$, then $\widehat{\Lambda_g} = \Lambda_{F(g)}$, where F is the extension of the Fourier transform from Plancherel's theorem.*

(b) *If Λ is a tempered distribution on $\mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, then*

  - *$\widehat{D^\alpha \Lambda} = s_\alpha \widehat{\Lambda}$, where $s_\alpha(x) = (ix)^\alpha$, and*

  - *$D^\alpha \widehat{\Lambda} = \widehat{m_\alpha \Lambda}$, where $m_\alpha(x) = (-ix)^\alpha$.*

(c) *The Fourier transform $\mathscr{F}$ of tempered distributions is an isomorphism of the space $(\mathscr{S}_d^*, w^*)$ onto itself. The following holds: $\mathscr{F}^4 = \mathrm{Id}$.*

III. The Bochner integral

1. Measurable mappings

Proposition 131. *Let Ω be a measurable space and X a metric space. Then the pointwise limit of a sequence of measurable mappings from Ω to X is a measurable mapping.*

Definition 132. Let Ω and X be sets. A mapping $f : \Omega \to X$ is called *simple* if $f(\Omega)$ is a finite set.

Theorem 133. *Let Ω be a measurable space and X a separable metric space. Then $f : \Omega \to X$ is measurable if and only if it is a pointwise limit of a sequence of simple measurable mappings from Ω to X.*

If (Ω, μ) is a measure space and M a set, then the symbol $\text{Æ}(\Omega, M)$ denotes the set of all mappings defined μ-a. e. on Ω with values in M.

Definition 134 (Salomon Bochner (1933))**.** Let (Ω, μ) be a measure space and X a metric space. A mapping from $\text{Æ}(\Omega, X)$ is called *strongly measurable* (or *Bochner measurable*) with respect to μ if it is a μ-a. e. pointwise limit of a sequence of simple measurable mappings from Ω to X.

Lemma 135. *Let (Ω, μ) be a space with a complete measure, X a metric space, and $f \in \text{Æ}(\Omega, X)$. Then f is strongly measurable if and only if it is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.*

Corollary 136. *Let (Ω, μ) be a space with a complete measure, X a metric space, and $\{f_n\} \subset \text{Æ}(\Omega, X)$ a sequence of strongly measurable mappings that converges pointwise a. e. to $f \in \text{Æ}(\Omega, X)$. Then f is strongly measurable.*

Lemma 137. *Let Ω and X be measurable spaces. Assume that X is also a vector space over $\mathbb{K}$, $f, g : \Omega \to X$ are simple measurable mappings, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are simple measurable mappings.*

Corollary 138. *Let (Ω, μ) be a measure space, X a normed linear space over $\mathbb{K}$, $f, g \in \text{Æ}(\Omega, X)$ strongly measurable, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are strongly measurable mappings.*

Definition 139 (Izrail Moisejevič Gelfand (1938), Billy James Pettis (1938))**.** Let (Ω, μ) be a measure space and X a normed linear space. A mapping $f \in \text{Æ}(\Omega, X)$ s called *weakly measurable* if $\phi \circ f$ is a measurable function for every $\phi \in X^*$.

Definition 140. Let X be a normed linear space. We say that $A \subset B_{X^*}$ is 1-*norming* if $\|x\| = \sup_{f \in A} |f(x)|$ for every $x \in X$.

Lemma 141. *Let X be a normed linear space. If $A \subset B_{X^*}$ is w^*-dense in B_{X^*}, then A is 1-norming.*

Lemma 142. *Let X be a normed linear space and let $A \subset B_{X^*}$ be 1-norming. Then $A_\circ = B_X$. More generally, $B(x, r) = \bigcap_{f \in A}\{y \in X; \; |f(y) - f(x)| \le r\}$ for every $x \in X$ and $r > 0$.*

Lemma 143. *Let X be a normed linear space and $A \subset X$. Then $\text{span}_{\mathbb{Q}} A$ is dense in $\text{span } A$ and $B_X \cap \text{span}_{\mathbb{Q}} A$ is dense in $B_X \cap \text{span } A$. It follows that if $M \subset X$ is separable, then $\overline{\text{span}} M$ is also separable.*

Theorem 144. *Let (Ω, μ) be a space with a complete measure, X a normed linear space, and $f \in \text{Æ}(\Omega, X)$. The following statements are equivalent:*

  (i) f is strongly measurable.

  (ii) f is measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.

  (iii) f is weakly measurable and there exists $E \subset \Omega$ such that $\mu(E) = 0$ and $f(\Omega \setminus E)$ is separable.

  (iv) There exist $E \subset \Omega$, $Y \subset X$ a separable subspace, and $A \subset B_{Y^*}$ countable such that $\mu(E) = 0$, $f(\Omega \setminus E) \subset Y$, $B_{Y^*} \cap \text{span } A$ is w^*-dense in B_{Y^*}, and $\phi \circ f$ is measurable for each $\phi \in A$.

Proposition 145. *Let (Ω, μ) be a space with a complete measure, X a normed linear space, and let $f \in \text{Æ}(\Omega, X)$ be strongly measurable. If $\phi \circ f = 0$ a. e. for every $\phi \in X^*$, then $f = 0$ a. e.*

2. The Bochner integral

Definition 146. Let (Ω, μ) be a measure space and X a normed linear space. A mapping $f : \Omega \to X$ is called a *step mapping* if it is simple, measurable, and $\mu(f^{-1}(x)) < +\infty$ for each $x \in f(\Omega) \setminus \{0\}$.

Definition 147. Let (Ω, μ) be a measure space, X a normed linear space, and $f : \Omega \to X$ a step mapping. Then for each measurable $E \subset \Omega$ we define the *Bochner integral* of f over E as

$$\int_E f \, \mathrm{d}\mu = \sum_{x \in f(\Omega) \setminus \{0\}} \mu(f^{-1}(x) \cap E)x.$$

Lemma 148. *Let (Ω, μ) be a measure space, X a normed linear space, and $f: \Omega \to X$ a step mapping. If $A, B \subset \Omega$ are disjoint measurable sets, then $\int_{A \cup B} f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu + \int_B f \, \mathrm{d}\mu$.*

Theorem 149. *Let (Ω, μ) be a measure space, X a normed linear space over $\mathbb{K}$, $f, g: \Omega \to X$ step mappings, and $\alpha \in \mathbb{K}$. Then $f + g$ and αf are step mappings and $\int_E (f + g) \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu + \int_E g \, \mathrm{d}\mu$ and $\int_E \alpha f \, \mathrm{d}\mu = \alpha \int_E f \, \mathrm{d}\mu$ for every meaurable $E \subset \Omega$.*

If (Ω, μ) is a measure space, X a normed linear space, and $f \in \text{\AE}(\Omega, X)$ a measurable mapping, then the function $t \mapsto \| f(t) \|$ is measurable on Ω, since it is a composition of a continuous function $\| \cdot \|$ with a measurable mapping f. This function will be denoted by $\| f \|$.

Lemma 150. *Let (Ω, μ) be a measure space, X a normed linear space, and $f: \Omega \to X$ a simple measurable mapping. Then f is a step mapping if and only if $\int_\Omega \| f \| \, \mathrm{d}\mu < +\infty$. In this case $\left\| \int_E f \, \mathrm{d}\mu \right\| \leq \int_E \| f \| \, \mathrm{d}\mu$ for every measurable $E \subset \Omega$.*

Lemma 151. *Let (Ω, μ) be a measure space, X a Banach space, $f \in \text{\AE}(\Omega, X)$, and let $f_n: \Omega \to X$, $n \in \mathbb{N}$ be a sequence of step mappings such that $\lim_{n \to \infty} \int_\Omega \| f_n(t) - f(t) \| \, \mathrm{d}\mu(t) = 0$. Then $\lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu$ exists for every measurable $E \subset \Omega$. Moreover, if $g_n: \Omega \to X$, $n \in \mathbb{N}$ is a sequence of step mappings with the same property as $\{f_n\}$, then $\lim_{n \to \infty} \int_E g_n \, \mathrm{d}\mu = \lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu$.*

Definition 152. Let (Ω, μ) be a measure space, X a Banach space, and $f \in \text{\AE}(\Omega, X)$. We say that f is *Bochner integrable* if there exists a sequence $f_n: \Omega \to X$, $n \in \mathbb{N}$ of step mappings such that $\lim_{n \to \infty} \int_\Omega \| f_n - f \| \, \mathrm{d}\mu = 0$. For every measurable $E \subset \Omega$ we then define the *Bochner integral* of f over E as

$$\int_E f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_E f_n \, \mathrm{d}\mu.$$

Theorem 153. *Let (Ω, μ) be a space with a complete measure, X a Banach space, and $f \in \text{\AE}(\Omega, X)$. Then f is Bochner integrable if and only if it is strongly measurable and $\| f \|$ is Lebesgue integrable. In this case $\left\| \int_E f \, \mathrm{d}\mu \right\| \leq \int_E \| f \| \, \mathrm{d}\mu$ for every measurable $E \subset \Omega$.*

Theorem 154. *Let (Ω, μ) be a space with a complete measure, X a Banach space over $\mathbb{K}$, $f, g \in \text{\AE}(\Omega, X)$ Bochner integrable and $\alpha \in \mathbb{K}$. Then the mappings $f + g$ and αf are Bochner integrable and $\int_E (f + g) \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu + \int_E g \, \mathrm{d}\mu$ and $\int_E \alpha f \, \mathrm{d}\mu = \alpha \int_E f \, \mathrm{d}\mu$ for every measurable $E \subset \Omega$.*

Theorem 155 (dominated convergence). *Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $\{f_n\} \subset \text{\AE}(\Omega, X)$ be a sequence of strongly measurable mappings. Let $f \in \text{\AE}(\Omega, X)$ be such that $f_n \to f$ pointwise a. e. and let $g \in L_1(\mu)$ be such that for each $n \in \mathbb{N}$ we have $\| f_n(t) \| \leq g(t)$ for a. a. $t \in \Omega$. Then f_n and f are Bochner integrable and $\lim_{n \to \infty} \int_\Omega \| f_n - f \| \, \mathrm{d}\mu = 0$. In particular, $\int_\Omega f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu$.*

Theorem 156. *Let (Ω, μ) be a space with a finite complete measure, X a Banach space, and let $\{f_n\} \subset \text{\AE}(\Omega, X)$ be a sequence of strongly measurable mappings. Let $f \in \text{\AE}(\Omega, X)$ be Bochner integrable such that $f_n \to f$ uniformly a. e. on Ω. Then $\int_\Omega f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu$.*

Theorem 157 (absolute continuity of the Bochner integral). *Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in \text{\AE}(\Omega, X)$ be Bochner integrable. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\left\| \int_E f \, \mathrm{d}\mu \right\| < \varepsilon$ whenever $E \subset \Omega$ is such that $\mu(E) < \delta$.*

Theorem 158. *Let (Ω, μ) be a space with a complete measure, X and Y Banach spaces, and let $f \in \text{\AE}(\Omega, X)$ be Bochner integrable and $T \in \mathcal{L}(X, Y)$. Then $T \circ f$ is Bochner integrable and*

$$\int_E T \circ f \, \mathrm{d}\mu = T \left(\int_E f \, \mathrm{d}\mu \right)$$

for every measurable $E \subset \Omega$.

Fact 159. *Let (Ω, μ) be a space with a complete measure, X a Banach space, $x \in X$, and $f \in L_1(\mu)$. Then $\int_E f(t) x \, \mathrm{d}\mu(t) = \left(\int_E f \, \mathrm{d}\mu \right) x$ for every measurable $E \subset \Omega$.*

Theorem 160. *Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in \text{\AE}(\Omega, X)$ be Bochner integrable. If $\int_E f \, \mathrm{d}\mu = 0$ for each measurable $E \subset \Omega$, then $f = 0$ a. e.*

Theorem 161. *Let (Ω, μ) be a space with a complete measure, X a Banach space, and let $f \in \text{\AE}(\Omega, X)$ be Bochner integrable. Then*

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \in \overline{\text{conv}} \, f(E).$$

for each $E \subset \Omega$ of positive measure.

Theorem 162 (Fubini's theorem for the Bochner integral). *Let (Ω_1, μ_1) and (Ω_2, μ_2) be spaces with σ-finite complete measures and let ν be a completion of the product measure $\mu_1 \times \mu_2$. Let X be a Banach space and let $f \in \text{Æ}(\Omega_1 \times \Omega_2, X)$ be Bochner integrable with respect to ν. Then for μ_1-a. a. $s \in \Omega_1$ the mapping $t \mapsto f(s,t)$ is Bochner integrable on Ω_2, for μ_2-a. a. $t \in \Omega_2$ the mapping $s \mapsto f(s,t)$ is Bochner integrable on Ω_1; the mappings $\psi_1(s) = \int_{\Omega_2} f(s,t)\, d\mu_2(t)$ and $\psi_2(t) = \int_{\Omega_1} f(s,t)\, d\mu_1(s)$ defined a. e. on Ω_1, resp. Ω_2 are Bochner integrable and*

$$\int_{\Omega_1} \psi_1 \, d\mu_1 = \int_{\Omega_1 \times \Omega_2} f \, d\nu = \int_{\Omega_2} \psi_2 \, d\mu_2.$$

3. The Lebesgue-Bochner spaces

Definition 163. Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \le p \le \infty$. The symbol $L_p(\mu, X)$ denotes the set of all strongly measurable mappings from $\text{Æ}(\Omega, X)$ such that $\|f\| \in L_p(\mu)$, factorised by the equality μ-a. e.

Further, for $f \in L_p(\mu, X)$ we define $\|f\|_{L_p(\mu,X)} = \big\| t \mapsto \|f(t)\| \big\|_{L_p(\mu)}$.

Theorem 164. *Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \le p \le \infty$.*

(a) $L_p(\mu, X)$ is a Banach space with the norm $\|f\|_{L_p(\mu,X)}$.

(b) If X is a Hilbert space, then $L_2(\mu, X)$ is a Hilbert space with the scalar product

$$\langle f, g \rangle_{L_2(\mu,X)} = \int_\Omega \langle f(t), g(t) \rangle \, d\mu.$$

Theorem 165. *Let (Ω, μ) be a space with a complete measure, X a Banach space, and $1 \le p < \infty$.*

(a) The set of all step mappings from Ω to X is dense in $L_p(\mu, X)$.

(b) If X and $L_p(\mu)$ are separable, then $L_p(\mu, X)$ is also separable.

IV. Compact convex sets

Definition 166. Let C be a convex subset of a vector space. We say that $x \in C$ is an *extreme point* of the set C if x is not an inner point of any segment lying in C, i.e. if $u, v \in C$ and $x = \lambda u + (1 - \lambda)v$ for some $\lambda \in (0, 1)$, then $u = v$. The set of all extreme points of C is denoted by $\operatorname{ext} C$.

Definition 167. Let C be a convex subset of a real vector space X. An affine hyperplane $W \subset X$ is called a *supporting hyperplane* of the set C (at a point $x \in C$), if $W \cap C \ne \emptyset$ (resp. $x \in W \cap C$) and C lies completely in one of the half-spaces determined by W (i.e. there exist a non-zero linear form f on X and $\alpha \in \mathbb{R}$ such that $W = f^{-1}(\alpha)$ and $\sup_C f \le \alpha$).

Fact 168. *Let C be a convex set in a vector space X.*

(a) If $B \subset C$ is convex, then $B \cap \operatorname{ext} C \subset \operatorname{ext} B$.

(b) If Y is a vector space and $T : X \to Y$ is an affine mapping, then T preserves convex combinations. If T is one-to-one, then $\operatorname{ext} T(C) = T(\operatorname{ext} C)$.

(c) If X is real and $W \subset X$ is a supporting hyperplane of the set C, then $\operatorname{ext}(C \cap W) = W \cap \operatorname{ext} C$.

Theorem 169. *If C is a compact convex subset of $\mathbb{R}^n$, then each point of the set C is a convex combination of at most $n + 1$ extreme points of the set C. Therefore $C = \operatorname{conv} \operatorname{ext} C$.*

Lemma 170. *Let $C \subset \mathbb{R}^n$ be convex. The either $\operatorname{Int} C \ne \emptyset$, or C lies in some affine hyperplane in $\mathbb{R}^n$.*

Corollary 171. *If $A \subset \mathbb{R}^n$ and $x \in \operatorname{conv} A$, then there exists at most $(n + 1)$-element subset $B \subset A$ such that $x \in \operatorname{conv} B$.*

Corollary 172. *Let $K \subset \mathbb{R}^n$ be compact. Then $\operatorname{conv} K$ is also compact.*

Proposition 173. *Let X be a Fréchet space and let $K \subset X$ be compact. Then $\overline{\operatorname{conv}} K$ and $\overline{\operatorname{aconv}} K$ are compact.*

Definition 174. Let C be a convex subset of a vector space. We say that a non-empty $E \subset C$ is an *extreme subset* of C if no point of E is a non-trivial convex combination of points from C some of which lie outside of E, i.e. if $\lambda x + (1 - \lambda)y \in E$ for some $x, y \in C$ and $\lambda \in (0, 1)$, then $x, y \in E$.

Definition 175. Let C be a convex subset of a vector space. We say that a function $f : C \to \mathbb{R}$ is *convex* if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$ for every $x, y \in C$ and $\lambda \in [0, 1]$.

Lemma 176. *Let C be a convex subset of a vector space, let $f : C \to \mathbb{R}$ be convex, and let E be an extreme subset of C. Then the set of points at which f attains its maximum over E is either empty, or an extreme subset of C.*

Lemma 177. *Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space) and let $C \subset X$ be convex. Then every compact extreme subset of C contains an extreme point of C.*

Recall that a real function f on a topological space X is called upper semi-continuous if the set $\{x \in X;\ f(x) \geq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

Theorem 178 (Bauer's maximum principle)**.** *Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space), let $K \subset X$ be a non-empty compact convex set and $f : K \to \mathbb{R}$ an upper semi-continuous convex function. Then f attains its maximum over K in an extreme point of K.*

Theorem 179 (Kreĭn-Milman)**.** *Let X be a topological vector space such that X^* separates the points of X (e.g. a Hausdorff locally convex space) and let $K \subset X$ be compact and convex. Then $K = \overline{\operatorname{conv}}\,\operatorname{ext} K$.*