UNIQUINĚM GÂTEAUX SMOOTH APPROXIMATIONS ON \( c_0(\Gamma) \)

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ABSTRACT. Every Lipschitz mapping from \( c_0(\Gamma) \) into a Banach space \( Y \) can be uniformly approximated by Lipschitz mappings that are simultaneously uniformly Gâteaux smooth and \( C^\infty \)-Fréchet smooth.

The main result of this note is a construction of uniform approximations of Lipschitz mappings from \( c_0(\Gamma) \) into a Banach space \( Y \), by means of Lipschitz mappings that are also uniformly Gâteaux (UG) smooth and \( C^\infty \)-Fréchet differentiable. We also construct an equivalent UG and \( C^\infty \)-Fréchet smooth renorming of \( c_0(\Gamma) \). Finally, we construct an example of a convex, even, Lipschitz and UG-smooth separating function, such that the Minkowski functional of its sub-level set is not UG-smooth. The first two results answer problems posed in [FMZ]. An example of the implicit function method violating UG smoothness was lacking in the literature. Its existence is not surprising to the specialists, as the known constructions of UG renormings always take a detour around this otherwise standard method of obtaining smooth renormings.

Let us recall that all Banach spaces admitting a UG-smooth bump function are in particular weak Asplund spaces by a fundamental result of Preiss, Phelps and Namioka in [PPN] (see also [F] or [BL]). However, the additional uniformity of the derivatives leads to a considerably stronger theory, with several characterisations of Banach spaces admitting a UG bump function (or a renorming). In particular, a Banach space admits an equivalent UG renorming if and only if its dual ball is a uniform Eberlein-Clark smooth norm, i.e. every separable Banach space admits a UG renorming. Namely, if a separable Banach space admits a \( C^k \)-Fréchet smooth norm, then it admits also a norm which is simultaneously \( C^k \)-Fréchet smooth and UG. The techniques used in their paper are strongly separable in nature, which leads to the natural question of what happens in the general case. This is the source of the questions asked in [DGZ] and [F–Z]. To shed some light on the significance of our results, let us briefly summarise some of the more recent results concerning UG smoothness, defined below (for simplicity we assume that the domain is a whole Banach space \( X \)).

It is shown in [LV] that a continuous UG-smooth real function on a Banach space \( X \) is locally Lipschitz. Moreover, if the function is uniformly continuous (or bounded), then it is globally Lipschitz. Thus some uniformity (Lipschitz) condition is in some sense also necessary for a mapping to be UG approximable. Tang [T] has shown that the existence of a UG bump function on a Banach space implies the existence of an equivalent UG renorming (analogous statement for Gâteaux smooth bumps is false, see [H]), and used this fact to show that every convex function on such a space is uniformly approximable by convex and UG-smooth functions on bounded sets. The more general problem of approximating all Lipschitz functions seems to be still open.

One of the difficulties is that the standard approach to constructing smooth approximations by using smooth partitions of unity appears to be failing (loss of uniformity). In this regard let us mention that in the stronger uniformly Fréchet smooth case it was shown by John, Toruńczyk and Zizler [JTZ] that the UF-smooth partitions of unity always exist provided the space has a UF bump function. However, the existence of UF approximations of Lipschitz functions seems to be open.

In the separable setting, Fabian and Zizler [FZ] were able to combine the best Fréchet smoothness of the space in question together with the UG condition (recall that every separable Banach space has a UG renorming). Namely, if a separable Banach space admits a \( C^k \)-Fréchet smooth norm, then it admits also a norm which is simultaneously \( C^k \)-Fréchet smooth and UG.

For an arbitrary set \( A \), we denote its cardinality by \( |A| \). For \( n \in \mathbb{N} \), \( \lambda_n \) denotes the Lebesgue measure on \( \mathbb{R}^n \).

For a metric space \( (X, \rho) \), we denote \( B(x, r) = \{ y \in X; \rho(x, y) \leq r \} \) and \( U(x, r) = \{ y \in X; \rho(x, y) < r \} \) the closed and open ball in \( X \) centred at \( x \in X \) with radius \( r \geq 0 \). Let \( A \subseteq X \). A neighbourhood \( U \subseteq X \) of \( A \) is called an \( r \)-uniform neighbourhood if there is \( r > 0 \) such that \( \bigcup_{x \in A} U(x, r) \subseteq U \). A neighbourhood is called a uniform neighbourhood if it is \( r \)-uniform for some \( r > 0 \).

For \( F \subseteq X \) we denote the associated projection in \( c_0(\Gamma) \) by \( P_F \), i.e. \( P_F x = \sum_{\gamma \in F} c_\gamma(x)e_\gamma \) where \( x \in c_0(\Gamma) \). By \( c_{00}(\Gamma) \) we denote the linear subspace of \( c_0(\Gamma) \) consisting of finitely supported vectors. The canonical supremum norm on \( c_0(\Gamma) \) will be denoted by \( \| \cdot \| \).

Let \( X \) and \( Y \) be normed linear spaces, \( \Omega \subseteq X \) be open and \( f : \Omega \to Y \). We will denote the directional derivative of \( f \) at \( x \in \Omega \) in the direction \( h \in X \) by \( D_h f(x) = \lim_{t \to 0} \frac{1}{t} (f(x + th) - f(x)) \). If \( f \) is Gâteaux differentiable for all \( x \in \Omega \) (i.e. \( D_h f(x) \) exists for all \( h \in X \) and \( h \mapsto D_h f(x) \) is a bounded linear operator) and moreover for all fixed \( h \in X \) the limit defining \( D_h f(x) \) is uniform in \( x \in \Omega \) we say that \( f \) is uniformly Gâteaux differentiable (UG) on \( \Omega \).

We denote by \( C^\infty (X, Y) \) the space of all \( C^\infty \)-Fréchet smooth mappings from \( X \) into \( Y \).

We are now ready to formulate the main results of our note.

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Theorem 1. Let $\Gamma$ be an arbitrary set, $Y$ be a Banach space, $f : c_0(\Gamma) \to Y$ be an $L$-Lipschitz mapping, and let $\varepsilon > 0$. Then there is an $L$-Lipschitz mapping $g \in C^{\infty}(c_0(\Gamma), Y)$ which is uniformly Gâteaux differentiable and such that it satisfies

$$\sup_{\tau_0(\Gamma)} \| f(x) - g(x) \| \leq \varepsilon.$$

Theorem 2. For any set $\Gamma$, $c_0(\Gamma)$ admits an equivalent norm which is simultaneously $C^{\infty}$-smooth and uniformly Gâteaux differentiable.

Theorem 1 is an immediate consequence of the following theorem.

Theorem 3. Let $\Gamma$ be an arbitrary set, $Y$ be a Banach space, $M \subset c_0(\Gamma)$, $U \subset c_0(\Gamma)$ be a uniform neighbourhood of $M$, $f : U \to Y$ be a uniformly continuous mapping with modulus of continuity $\omega$ and let $\varepsilon > 0$. Then there is a mapping $g \in C^{\infty}(c_0(\Gamma), Y)$ which uniformly locally depends on finitely many coordinates such that $\sup_\Omega \| f(x) - g(x) \| \leq \varepsilon$ and $g$ is uniformly continuous on $M$ with modulus of continuity dominated by $\omega$. If $f$ is moreover $L$-Lipschitz, then $g$ is $L$-Lipschitz on $M$ and uniformly Gâteaux differentiable on Int $M$.

We will prove the theorems by using a few lemmata.

Lemma 4. Let $X, Y$ be normed linear spaces, $\Omega \subset X$ be open and $f : \Omega \to Y$ be a Gâteaux differentiable mapping. If for each $h \in X$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on $\Omega$, then $f$ is uniformly Gâteaux differentiable on $\Omega$ provided that $\Omega$ is convex; otherwise $f$ is uniformly Gâteaux differentiable on any open $V \subset \Omega$ satisfying $\text{dist}(V, X \setminus \Omega) > 0$. Conversely, if $f$ is uniformly Gâteaux differentiable and uniformly continuous on $\Omega$, then for each $h \in X$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on any $A \subset \Omega$ satisfying $\text{dist}(A, X \setminus \Omega) > 0$.

Proof. Choose $h \in X$ and $\varepsilon > 0$, and find $\delta > 0$ such that $\| D_h f(x + th) - D_h f(x) \| < \varepsilon$ for all $x \in \Omega$ and $t \in (-\delta, \delta)$ such that $x + th \in \Omega$. If $\Omega$ is convex we set $V = \Omega$ and $\eta = \delta$, otherwise let $\eta = \min\{|t|, \text{dist}(V, X \setminus \Omega)\}$. Fix $x \in V$ and define a mapping $g : I \to Y$ by $g(t) = f(x + th) - t D_h f(x)$, where $I = [t \in (-\eta, \eta) : x + th \in \Omega]$. Notice that $I$ is an open interval containing 0 and $g'(t) = D_h f(x + th) - D_h f(x)$ for $t \in I$. By the assumption, $\|g'(t)\| \leq \varepsilon$ for $t \in I$, hence $g$ is $\varepsilon$-Lipschitz on $I$, and so $\frac{1}{2} \| f(x + th) - f(x) - D_h f(x) \| = \frac{1}{2} \| g(t) - g(0) \| \leq \varepsilon$ for all $t \in I$.

To prove the converse statement, choose $h \in H$, $h \neq 0$, a subset $A \subset \Omega$ for which $\text{dist}(A, X \setminus \Omega) > 0$, and $\varepsilon > 0$. Find $0 < \eta < \text{dist}(A, X \setminus \Omega)/\|h\|$ such that $\| f(x + \eta h) - f(x)/\eta - D_h f(x) \| < \frac{\varepsilon}{2}$ for any $x \in A$. Let $\delta > 0$ be such that $\| f(x) - f(y) \| < \eta \frac{\varepsilon}{2}$ whenever $x, y \in A$ are such that $\| x - y \| < \delta$. Then, for such $x, y$, we have

$$\| D_h f(x) - D_h f(y) \| < \varepsilon \frac{1}{2} + \frac{\varepsilon}{\eta} \| f(x + \eta h) - f(x) - f(y + \eta h) + f(y) \| < \varepsilon.$$  

Lemma 5. Let $X$ and $Y$ be normed linear spaces, $H$ be a dense subset of $X$, $\Omega \subset X$ be open and $f : \Omega \to Y$ be a Gâteaux differentiable Lipschitz mapping such that for each $h \in H$ the mapping $x \mapsto D_h f(x)$ is uniformly continuous on $\Omega$. Then the mapping $x \mapsto D_h f(x)$ is uniformly continuous on $\Omega$ for every $h \in \Omega$.

Proof. Let $L$ be a Lipschitz constant of $f$. Pick an arbitrary $h \in X$ and let $\varepsilon > 0$. Find $h_0 \in H$ such that $\|h - h_0\| < \frac{\varepsilon}{2L}$. By the uniform continuity of $D_{h_0} f(x)$ there is $\delta > 0$ such that $\| D_{h_0} f(x) - D_{h_0} f(y) \| < \frac{\varepsilon}{2}$ whenever $x, y \in \Omega$, $\| x - y \| < \delta$. Then

$$\| D_h f(x) - D_h f(y) \| \leq \| D_{h_0} f(x) - D_{h_0} f(y) \| + \| D_{h_0} - D_{h_0} - h_0 \| \| x - y \| < \frac{\varepsilon}{2} + 2L \| h - h_0 \| < \varepsilon,$$

whenever $x, y \in \Omega$, $\| x - y \| < \delta$.

Lemma 6. Let $X$ be a normed linear space, $k \in \mathbb{N} \cup \{\infty\}$, $g : X \to \mathbb{R}$ be a $C^k$-smooth, UG, Lipschitz, even and convex function that is separating (i.e. there is an $r > 0$ such that $\inf_{x \in rS_X} |g(x) - g(0)| > 0$). Then $X$ admits an equivalent $C^k$-smooth UG norm.

Proof. As shown in the Example below, UG smoothness does not, in general, survive the standard use of the implicit function theorem (Minkowski functional). To be able to use the Minkowski functional of some sub-level set of $g$, we need to gain more control over $g'(x)$. To this end we introduce a transformation, the idea of which comes from [FZ]. Basically, we construct a function that is “primitive” to $g$ in a sense, so that its derivative is $g$ back again (more or less), hence Lipschitz. (For a more detailed exposition of the method we refer to [FZ].) So, define $f : X \to \mathbb{R}$ by

$$f(x) = \int_{[0,1]} g(tx) \, d\lambda(t).$$

Let $L$ be the Lipschitz constant of $g$. It is easy to check that $f$ is $L/2$-Lipschitz, even and convex.

Without loss of generality we may assume that $g(0) = 0$. By the convexity of $g$ and the fact that $g$ is even, $g(x) \geq 0$ for $x \in X$. Since $g$ is separating, there are $r > 0$ and $a > 0$ such that $g(x) \geq a$ for all $x \in rS_X$. Hence $g(tx) \geq a - Lr(1-t)$ whenever $t \in [0,1]$ and $\|x\| = r$. It follows that

$$f(x) \geq \int_{1-a/(1-Lr)}^1 (a - Lr(1-t)) \, d\lambda(t) = \frac{a^2}{2Lr} = b$$

for any $x \in rS_X$. (1)
Let $x \in X$. Using the compactness of the set $\{tx; t \in [0, 1]\}$ and the continuity of $g'$, we can find a neighbourhood $U$ of $x$ such that $g'(ty)$ is bounded for $y \in U$ and $t \in [0, 1]$. Hence, using the theory of integration, we can check that $f \in C^1(U)$ and

$$f'(x)h = \int_{[0,1]} g'(tx)(t h) \, d\lambda(t).$$

By repeating the same argument it follows that $f \in C^k(X)$.

Using Lemma 4 and 2 we can see that the function $x \mapsto f'(x)h$ is uniformly continuous on $X$ for any $h \in X$.

Moreover, the function $x \mapsto f'(x)x$ is Lipschitz on $X$. Indeed, using the substitution $t(1+\tau) = s$ we get $f(x + \tau x) = f(0) + g(t(x + \tau x)) \, d\lambda(t) = \frac{1}{1+\tau} \int_{[0,1]} g(sx) \, d\lambda(s)$. Thus, using the continuity of $g$ along the way,

$$f'(x)x = \lim_{\tau \to 0} \frac{1}{1+\tau} \int_{[0,1]} g(tx + \tau tx) \, d\lambda(t) - f(x) = \lim_{\tau \to 0} \frac{1}{1+\tau} \left( \frac{1}{1+\tau} - 1 \right) \int_{0}^{1+\tau} g(tx) \, d\lambda(t) + \int_{0}^{1} g(tx) \, d\lambda(t)$$

$$= \lim_{\tau \to 0} \frac{-1}{1+\tau} \int_{0}^{1+\tau} g(tx) \, d\lambda(t) + \lim_{\tau \to 0} \frac{1}{1} \int_{0}^{1} g(tx) \, d\lambda(t) = g(x) - \int_{0}^{1} g(tx) \, d\lambda(t) = g(x) - f(x).$$

Since both $f$ and $g$ are $L$-Lipschitz, the function $x \mapsto f'(x)x$ is $2L$-Lipschitz. Clearly, $f(0) = 0$. So, the convexity of $f$ implies

$$f'(x)x \geq f(x) \quad \text{for any } x \in X. \quad (3)$$

Using the convexity of $f$, the estimate (1) and the fact that $f(0) = 0$, we obtain that $f(x) > b$ whenever $\|x\| > r$. It follows that $B = \{x \in X; f(x) \leq b\}$ is a closed absolutely convex bounded set that contains a neighbourhood of $0$. The Minkowski functional $v$ of the set $B$ is therefore an equivalent norm on $X$ and $v(x) = 1$ if and only if $f(x) = b$.

Put $M = X \times (0, +\infty)$ and define $F: M \to \mathbb{R}$ by $F(x,y) = f(\frac{x}{y}) - b$. It is obvious that $F \in C^k(M)$. Pick any $0 \neq x \in X$. Then $(x, v(x)) \in M$, $F(x, v(x)) = 0$, and

$$\frac{\partial F}{\partial y}(x, v(x)) = f'(\frac{x}{v(x)}) \left( \frac{x}{v(x)} \right)^2 = -\frac{v(x)}{v(x)} f'(\frac{x}{v(x)}) \left( \frac{x}{v(x)} \right).$$

From (3) it follows that $\frac{\partial F}{\partial y}(x, v(x)) \leq -b/v(x)$. Therefore we can use the Implicit Function Theorem to conclude that on some neighbourhood of $x, v$ is a $C^k$-smooth function. Hence $v$ is a $C^k$-smooth norm.

Finally, we claim that the function $x \mapsto v'(x)h$ is uniformly continuous on $A_R = X \setminus B(0, R)$ for any $h \in X$ and any $R > 0$, which according to Lemma 4 means that the norm $v$ is UG.

Since $F(x, v(x)) = 0$ for any $0 \neq x \in X$, it follows that $(F(x, v(x)))' = 0$. A simple computation yields

$$v'(x) = \frac{f'(\frac{x}{v(x)})}{f'(\frac{x}{v(x)}) \left( \frac{x}{v(x)} \right)^2}.$$
Proof. Let \( \eta > 0 \) be such that \( \Omega \) is an \( \eta \)-uniform neighbourhood of \( M \) and find \( 0 < \delta < \min\{ \eta, \frac{\epsilon}{2} \} \) such that \( \omega(\delta) < \epsilon \). Choose \( \varphi \) to be an even \( C^\infty \)-smooth non-negative function on \( \mathbb{R} \) such that \( \text{supp} \varphi \subseteq [-\delta, \delta] \) and \( \int_{\mathbb{R}} \varphi = 1 \). We denote \( C = \int_{\mathbb{R}} |\varphi(t)| \, dt \).

Let \( \mathcal{F} \subset 2^{\Gamma} \) be a partially ordered set of non-empty finite subsets of \( \Gamma \), ordered by inclusion. For any \( F \in \mathcal{F} \), we define the mapping \( \Psi_F : c_0(\Gamma) \to \mathbb{R} \) by

\[
\Psi_F(x) = \int_{\mathbb{R}[\Gamma]} \Phi\left(x - \sum_{y \in F} t_y e_y\right) \prod_{y \in F} \varphi(t_y) \, d\lambda_{|F|(t)},
\]

where we integrate in the Bochner sense. Notice, that the integral is well-defined, since \( \Phi = 0 \) on the closed set \( c_0(\Gamma) \setminus \Omega \) and \( \Phi \) is uniformly continuous on \( \Omega \).

The net \( \{\Psi_F\}_{F \in \mathcal{F}} \) converges on \( c_0(\Gamma) \) to a mapping \( \Psi : c_0(\Gamma) \to \mathbb{R} \). In fact, we claim that for any \( x \in c_0(\Gamma) \), there is an \( F \in \mathcal{F} \) such that \( \Psi_F(x) = \Psi_F(y) \) for any \( F \subset H \in \mathcal{F} \) and any \( y \in U(x, \frac{\epsilon}{2}) \). Indeed, for a fixed \( x \in c_0(\Gamma) \) let \( F \in \mathcal{F} \) be such that \( \Phi \) depends only on \( \{e_y\}_{y \in F} \) on \( U(x, r) \) and \( \|x - P_F y\| < \frac{\epsilon}{2} \). Choose any \( y \in U(x, \frac{\epsilon}{2}) \) and \( H \supseteq F \). Suppose that \( t_y \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \) for all \( y \in H \). Then \( \|x - \left(\sum_{y \in H} t_y e_y\right)\| < r \) and consequently \( \Phi\left(y - \sum_{y \in H} t_y e_y\right) = \Phi\left(y - \sum_{y \in F} t_y e_y\right) \).

Thus, by Fubini’s theorem,

\[
\Psi_H(y) = \int_{[-\delta, \delta]^H} \Phi\left(y - \sum_{H \ni F} t_y e_y\right) \prod_{y \in H} \varphi(t_y) \, d\lambda_{|H|(t)} = \int_{[-\delta, \delta]^F} \Phi\left(y - \sum_{F \ni K} t_y e_y\right) \prod_{y \in F} \varphi(t_y) \, d\lambda_{|F|(t)}.
\]

Moreover, \( \|x - P_F y\| \leq \|x - P_F x\| + \|P_F y - y\| < r \) and so we can easily see that \( \Psi_F(y) = \Psi_F(P_F y) \). The mapping \( \Psi_{|F \cap \Gamma(\Gamma')} \) is in fact a finite-dimensional convolution with a smooth kernel on \( \mathbb{R}[\Gamma] \), and so \( \Psi_F \) is a \( C^\infty \)-smooth mapping on \( U(x, \frac{\epsilon}{2}) \).

The mapping \( \Psi \) is therefore \( U(0, \frac{\epsilon}{2}) \)-ULFC-{\( e_y \}_{y \in \Gamma} \} and \( \Psi \in C^\infty(c_0(\Gamma), Y) \), as for any \( x \in c_0(\Gamma) \), \( \Psi(x) = \Psi(F \circ P_F) \) on \( U(x, \frac{\epsilon}{2}) \) for some \( F \in \mathcal{F} \). To show that \( \sup \|F\| \cdot \|\Phi(x) - \Psi(x)\| \leq \epsilon \) choose any \( x \in M \subset \Omega \). Let \( F \in \mathcal{F} \) be such that \( \Psi(x) = \Psi(F) \). Notice that \( \|x - \left(\sum_{y \in F} t_y e_y\right)\| \leq \sum_{y \in F} \|t_y e_y\| \leq \delta < \eta \) whenever \( t_y \in [-\delta, \delta] \) for all \( y \in F \). Hence \( x - \sum_{y \in F} t_y e_y \in \Omega \) and

\[
\|\Phi(x) - \Psi(x)\| = \|\Phi(x) - \Phi(F)\| = \left| \int_{\mathbb{R}[\Gamma]} \Phi(x) \prod_{y \in F} \varphi(t_y) \, d\lambda_{|F|(t)} - \int_{\mathbb{R}[\Gamma]} \Phi(x - \sum_{y \in F} t_y e_y) \prod_{y \in F} \varphi(t_y) \, d\lambda_{|F|(t)} \right| \leq \int_{[-\delta, \delta]^F} \|\Phi(x) - \Phi(x - \sum_{y \in F} t_y e_y)\| \prod_{y \in F} \varphi(t_y) \, d\lambda_{|F|(t)} \leq \int_{[-\delta, \delta]^F} \left| \Phi(x - \sum_{y \in K} t_y e_y) - \Phi(y - \sum_{y \in K} t_y e_y) \right| \prod_{y \in K} \varphi(t_y) \, d\lambda_{|K|(t)} \leq \omega(\delta) < \epsilon.
\]

To see that the modulus of continuity of \( \Psi \) on \( M \) is dominated by \( \omega \), choose \( x, y \in M \) and find \( F, H \in \mathcal{F} \) such that \( \Psi(x) = \Psi(F) \) and \( \Psi(y) = \Psi_H(y) \). Then for \( K = F \cup H \) we have \( \Psi(x) = \Psi_K(x) \) and \( \Psi(y) = \Psi_K(y) \). As \( x - \sum_{y \in K} t_y e_y \in \Omega \) and \( y - \sum_{y \in K} t_y e_y \in \Omega \) whenever \( t_y \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \) for all \( y \in K \),

\[
\|\Psi(x) - \Psi(y)\| = \|\Psi_K(x) - \Psi_K(y)\| \leq \int_{[-\delta, \delta]^K} \left| \Phi(x - \sum_{y \in K} t_y e_y) - \Phi(y - \sum_{y \in K} t_y e_y) \right| \prod_{t \in [a]} \varphi(t) \, d\lambda_{|K|(t)} \leq \omega(\|x - y\|).
\]

Similarly we can check that \( \Psi \) is even if \( \Phi \) is even and \( \Psi \) is convex under the additional assumptions that \( Y = \mathbb{R} \) and \( \Phi \) is convex.

We finish the proof by showing that the directional derivatives of \( \Psi \) in the directions of \( c_0(\Gamma) \) are uniformly continuous on \( M \). So first, choose any \( a \in \Gamma \). For \( x, y \in M \) find \( F, H \in \mathcal{F} \) such that \( \Psi(x) = \Psi_F(x) \) on \( U(x, \frac{\epsilon}{2}) \) and \( \Psi(y) = \Psi_H(y) \) on \( U(y, \frac{\epsilon}{2}) \). Put \( K = F \cup H \cup \{a\} \). It is standard to show that

\[
\Psi_K'(x) e_a = \int_{\mathbb{R}[\Gamma]} \Phi\left(x - \sum_{y \in K} t_y e_y\right) \prod_{y \in K \setminus \{a\}} \varphi(t_y) \, d\lambda_{|K|(t)}.
\]

Hence, similarly as above,

\[
\left|\Psi'(x) e_a - \Psi'(y) e_a\right| \leq \int_{\mathbb{R}[\Gamma]} \left|\Phi\left(x - \sum_{y \in K} t_y e_y\right) - \Phi\left(y - \sum_{y \in K} t_y e_y\right)\right| \prod_{y \in K \setminus \{a\}} \varphi(t_y) \, d\lambda_{|K|(t)} \leq \omega(\|x - y\|) \int_{\mathbb{R}} |\varphi'(t)| \, dt = C \omega(\|x - y\|).
\]

Next, choose any \( h \in c_0(\Gamma) \) and \( x, y \in M \). It follows from (4) that

\[
\left|\Psi'(x) h - \Psi'(y) h\right| \leq \sum_{y \in \text{supp} h} \left|\Psi'(x) (e_y^*(h) e_y) - \Psi'(y) (e_y^*(h) e_y)\right| \leq C \omega(\|x - y\|) \sum_{y \in \text{supp} h} |e_y^*(h)| = C \|h\|_{\ell_1} \omega(\|x - y\|).
\]

\( \Box \)
Proof of Theorem 2. Without loss of generality we may assume that $U$ is open. Let $r > 0$ be such that $U$ is an $r$-uniform open neighbourhood of $M$ and find $0 < \eta \leq \frac{r}{2}$ such that $\omega(\eta) < \frac{\eta}{2}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \max\{0, t - \eta\} + \min\{0, t + \eta\}$. Then $\varphi$ is 1-Lipschitz and $|\varphi(t) - t| \leq \eta$ for all $t \in \mathbb{R}$.

Further, define a mapping $\phi : c_0(\Gamma) \to c_0(\Gamma)$ by $\phi(x) = \sum_{y \in E} \varphi(e^*_y(x)) e_y$. (Notice that in fact $\phi$ maps into $c_0(\Gamma)$.) Then $\phi$ is 1-Lipschitz and $|\phi(x) - x| \leq \eta$ for all $x \in c_0(\Gamma)$. Moreover, we claim that $\phi$ is $U(0, \frac{\eta}{2})$-ULFC-$\{e^*_y\}_{y \in \Gamma}$. Indeed, fix $x \in c_0(\Gamma)$ and find $F \subset \Gamma$ such that $|F| < \infty$ such that $\|x - P_F x\| < \frac{\eta}{2}$. Then for any $y \in U(x, \frac{\eta}{2})$ we have $\|y - P_F y\| < \eta$. This means that if $y, z \in U(x, \frac{\eta}{2})$ are such that $e^*_y(y) = e^*_z(z)$ for all $y \in F$, then $\varphi(e^*_y(y)) = 0 = \varphi(e^*_z(z))$ for all $y \in \Gamma \setminus F$ and hence $\varphi(e^*_y(y)) = \varphi(e^*_z(z))$ for all $y \in F$. Hence $\phi(y) = \phi(z)$, and so $\phi$ depends only on $\{e^*_y\}_{y \in \Gamma}$. Then $\phi(x) = \phi(y)$ and hence $\|\phi(x) - \phi(y)\| \leq \omega(\|x - y\|)$. Moreover, we claim that $\phi(y) \in U(y, \frac{\eta}{2})$. Therefore, $\|\phi(y) - y\| < \frac{\eta}{2}$.

The function $\phi(x)$ is well defined for all $x \in c_0(\Gamma)$ and hence $\|\phi(x) - \phi(y)\| \leq \omega(\|x - y\|)$.

Finally, $\sup_n \|f_n(x) - \phi(x)\| \leq \sup_n \|\phi(x) - \phi(y)\| \leq \omega(\|x - y\|)$.

Lemma 7 together with Lemma 5 and Lemma 4 finishes the proof.

\[ \square \]

Proof of Theorem 2. Define a function $\Phi : c_0(\Gamma) \to \mathbb{R}$ by $\Phi(x) = \max\{0, \|x\| - 1\}$. Then $\Phi$ is a 1-Lipschitz convex function which is $U(0, \frac{1}{2})$-ULFC-$\{e^*_y\}_{y \in \Gamma}$. (Notice that $\Phi = \|\cdot\| \circ \phi$ as in the proof of Theorem 3 for $\eta = 1$.)

Let $g \in C^\infty(c_0(\Gamma))$ be a 1-Lipschitz convex function with uniformly continuous directional derivatives produced by Lemma 7 combined with Lemma 5 such that $|g(x) - \Phi(x)| \leq 1$ for all $x \in c_0(\Gamma)$. Then $g$ is separating, as $g(0) \leq 1$ and $g(x) \geq 2$ on $45\cdot X$. The function $g$ is also UG by Lemma 4 and so we can finish by using Lemma 6.

\[ \square \]

The technique used in the above proof can be used to strengthen the main result in [HZ] on the existence of $C^\infty$-Fréchet smooth approximations of strongly lattice norms on $c_0(\Gamma)$, by placing the additional UG smoothness requirement. We prefer to omit the details of the proof.

Example. We will sketch a construction of a UG, Lipschitz, even and convex function on $f$ that is separating, but the Minkowski functional of its sub-level set is an equivalent norm on $c_0$ that is not UG-smooth. The existence of such examples was suspected by specialists (e.g. [FZ, T]), but no explicit construction seems to be available in the literature.

For any $n \in \mathbb{N}$, let $f_n : c_0 \to \mathbb{R}$ and $g_n : c_0 \to \mathbb{R}$ be defined as

\[
  f_n = e^*_1 + e^*_2 - e^*_n + 1,
  g_n = e^*_1 + 2 - \frac{1}{2n} e^*_2 + e^*_n - 1.
\]

Let $f_n = f_n - e^*_2/4$, $g_n = g_n - e^*_2/4$, and further $g(x) = \sup_{n} f_n(x)$, $g_n(x)$, $\|x\|/4$ and $f(x) = g(x) + g(-x)$. It is easy to see that $f(0) = 0$, $f$ is 8-Lipschitz, even, convex and separating. Further, for all $n \in \mathbb{N}$ and $w \in c_0$, let

\[
  x_n = e^*_2n, \quad y_n = e^*_2n + 1 - \frac{1}{2n} e^*_2n + 1.
  U_n(w) = \left\{ \begin{array}{l}
  x \in c_0 : |e^*_2n(x - w)| \leq \frac{1}{16n}, |e^*_2n+1(x - w)| \leq \frac{1}{16n}, |e^*_2n(x - w)| < \frac{1}{16} \end{array} \right\}.
\]

We can check that $g(x) = f_n(x)$ on $U_n(x_n)$, $g(x) = g_n(x)$ on $U_n(y_n)$, and $g(-x) = \|x\|/4 = e^*_2n(x)/4$ on both $U_n(x_n)$ and $U_n(y_n)$. Hence, $f = f_n$ on $U_n(x_n)$ and $f = g_n$ on $U_n(y_n)$.

Now, for each $n \in \mathbb{N}$ let $\psi_n$ be an even $C^\infty$-smooth non-negative function on $\mathbb{R}$ such that $\int_{\mathbb{R}} \psi_n = 1$. Moreover, choose these functions so that $\sup \psi_n \subset \left[ -\frac{1}{2}, \frac{1}{2} \right]$, $\sup \psi_{2n} \subset \left[ -\frac{1}{2n}, \frac{1}{2n} \right]$ and $\sup \psi_{2n+1} \subset \left[ -\frac{1}{2n}, \frac{1}{2n} \right]$ for $n \in \mathbb{N}$. Similarly as in [FZ] or [T] we can show that the function

\[
  F(x) = \lim_{n \to \infty} \int_{\mathbb{R}} f \left( x - \sum_{j=1}^{n} t_j e_j \right) \prod_{j=1}^{n} \psi_j(t_j) d \lambda_n(t)
\]

is well defined for all $x \in c_0$ and that it is Lipschitz, even, convex, separating and UG.

Furthermore, notice that for each $n \in \mathbb{N}$, $f$ is affine on both $U_n(x_n)$ and $U_n(y_n)$. Since the convolution of an affine function with an even kernel is the same affine function again, there are neighbourhoods of $x_n$ and $y_n$ such that $F = f_n$ (or $F = g_n$ respectively) on those neighbourhoods.
Let \( v \) be the Minkowski functional of the set \( B = \{ x \in c_0; \ F(x) \leq 1 \} \). Then \( v(x_n) = 1 = v(y_n) \) for all \( n \in \mathbb{N} \),
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0,
\]
but
\[
v'(x_n)e_1 = \frac{F'(x_n)e_1}{F'(x_n)x_n} = \frac{f'_n(x_n)e_1}{f'_n(x_n)x_n} = \frac{1}{1} = 1 \quad \text{and} \quad v'(y_n)e_1 = \frac{F'(y_n)e_1}{F'(y_n)y_n} = \frac{g'_n(y_n)e_1}{g'_n(y_n)y_n} = \frac{1}{2}.
\]

**REFERENCES**


