A REMARK ON SMOOTH IMAGES OF BANACH SPACES

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ABSTRACT. Let $X$ be a Banach space with a non-separable super-reflexive quotient. Then for any separable Banach space $Y$ of dimension at least two there exists a $C^\infty$-smooth surjective mapping $f : X \to Y$ such that the restriction of $f$ onto any separable subspace of $X$ fails to be surjective. This solves a problem posed by Aron, Jaramillo, and Ransford (Problem 186 in the book [GMZ]).

1. Smooth images

It was shown by Bates [3] that every separable Banach space $Y$ is a range of a $C^1$-smooth surjection $f : X \to Y$ from any infinite dimensional separable Banach space $X$. Moreover, under rather general conditions $f$ can be chosen to be $C^\infty$-smooth. On the other hand, it was shown in [H3] that if $X = c_0$ and $Y = \ell_2$, then $f$ cannot be $C^2$-smooth. In the setting of the non-separable space $X = c_0(\omega_1)$ it turns out ([GHM]) that the existence of $C^2$-smooth surjections onto $\ell_2$ depends on the additional axioms of set theory. In particular, under the continuum hypothesis such surjections easily exist, while under Martin’s axiom MA$_{\omega_1}$ (which implies the negation of the continuum hypothesis) there are no such $C^2$-smooth surjections.

In a recent paper of Aron, Jaramillo, and Ransford, [AJR], the following result was shown: Let $\Gamma$ be a set of cardinality at least continuum $\mathfrak{c}$, and suppose there exists a bounded linear operator $L : X \to c_0(\Gamma)$, such that $L(X)$ contains the canonical basis of $c_0(\Gamma)$. Then for any separable Banach space $Y$ of dimension at least two there exists a $C^\infty$-smooth surjective mapping $f : X \to Y$ such that the restriction of $f$ onto any separable subspace of $X$ fails to be surjective.

Of course, the result applies in particular to spaces $X = c_0(\omega) = \ell_p(\omega)$, $1 \leq p < \infty$. As we have seen above, this result holds for $c_0(\omega_1)$ if we assume the continuum hypothesis, but it fails under MA$_{\omega_1}$, as $\ell_2$ cannot be a range of a $C^\infty$-smooth surjection. It is therefore quite natural to ask what happens for $X = \ell_p(\omega_1)$ in this respect. The authors in [AJR] speculate that the result perhaps again depends on additional axioms of set theory. The problem is posed also in the recent monograph of Guirao, Montesinos, and Zizler, [GMZ], Problem 186.

Corollary 5 in this note gives a solution to this problem. In particular, there is a $C^\infty$-smooth surjection $f : \ell_p(\omega_1) \to Y$ onto any separable Banach space $Y$ of dimension at least 2 such that the restriction of $f$ onto any separable subspace of $\ell_p(\omega_1)$ fails to be surjective. Our result is in fact somewhat more general, and applies in particular to all non-separable super-reflexive Banach spaces.

Before formulating the main theorem we recall some basic definitions on trees. A tree is a partially ordered set $(T, \preceq)$ with the property that for every $t \in T$ the subset $\{s \in T : s \preceq t\}$ is well-ordered. For $t \in T$ we denote by $t^+$ the set of all immediate successors of $t$, i.e., $t^+ = \{u \in T : s < u \text{ if and only if } s \preceq t\}$. For $u \in T$ we write $u^-$ for the unique $t \in T$ such that $t < u^+$, if such $t$ exists. If a tree has a least element, then we will call this tree rooted. We will assume that the least element of a rooted tree is designated by 0, unless stated otherwise. The height of an element $t \in T$ is a unique ordinal $ht(t)$ with the same order type as $\{s \in T : s < t\}$. The height of the tree $T$ is defined by $sup\{ht(t) + 1 : t \in T\}$. A branch of $T$ is a maximal linearly ordered subset and we denote by $\mathcal{B}(T)$ the set of all branches of $T$. For an ordinal $\alpha$ we denote by $T_{\alpha} = \{t \in T : ht(t) = \alpha\}$ the $\alpha$th level of the tree $T$. For a branch $b \in \mathcal{B}(T)$ we denote $b_\alpha = b \cap T_{\alpha}$. Let $\mu$ be a cardinal. We say that $T$ is $\mu$-branching if card $T_0 \leq \mu$ and card $t^+ \leq \mu$ for each $t \in T$.

Let $\mu$ be a cardinal. We say that a subset $S$ of a topological space $X$ is $\mu$-Suslin in $X$ if there is a $\mu$-branching tree $T$ of height $\omega$ and closed sets $F_t \subset X, t \in T$ such that $S = \bigcup_{b \in \mathcal{B}(T)} \bigcap_{t \in \mu} F_{b_t}$. We remark that $\omega$-Suslin sets are called simply Suslin in the classical terminology and that a classical result states that in Polish spaces Suslin sets (our $\omega$-Suslin sets) are precisely the analytic sets, see e.g. [K] Theorem 25.7.

Some more notation: By $B(x, r)$, resp. $U(x, r)$ we denote the closed, resp. open ball in a metric space centred at $x$ and with radius $r$. By $\mathcal{L}(X; Y)$, resp. $\mathcal{P}(X; Y)$ we denote the space of continuous linear operators, resp. continuous $n$-homogeneous polynomials from $X$ to $Y$; for an introduction to the theory of polynomials see e.g. [HI] Chapter 1. If $f$ is a mapping into a vector space $Y$, then $sup_{x \in X} f = f^{-1}(Y \setminus \{0\})$. If $x, y \in Y$, then $[x, y]$ denotes the segment with endpoints $x$ and $y$.

Our main result is the following:

Theorem 1. Let $X$ be an infinite-dimensional Banach space that admits a $C^k$-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$, with each derivative bounded on $X$. Let $Y$ be a Banach space with $dens Y \leq dens X$, let $C \subset Y$ be convex, $y_1 \in C$, and $C \subset A \subset C$ a dens $Y$-Suslin set. Then there is $f \in C^k(X; Y)$ with $supp f \subset B_X$ such that $f(X) = [0, y_1] \cup A$. 

Date: March 2017.
2010 Mathematics Subject Classification. 46B26, 46B80, 46T20.
Key words and phrases: smooth surjections.
Supported by GAČR 16-07378S.
Before proving the theorem we show some of its corollaries. For this we also need the following theorem of Felix Hausdorff, [Ha Satz I], see also [I] Exercise 29.8).

**Theorem 2.** Every uncountable Polish space is a union of an increasing \( \omega_1 \)-sequence of \( G_\delta \) sets.

**Corollary 3.** Let \( X \) be a non-separable Banach space that admits a \( C^k \)-smooth bump, \( k \in \mathbb{N} \cup \{\infty\} \) with each derivative bounded on \( X \), and let \( Y \) be a separable Banach space with \( \dim Y \geq 2 \). Then there is \( f \in C^k(X; Y) \) such that \( f(X) = Y \) but \( f(Z) \neq Y \) for any separable subset \( Z \subset X \).

**Proof.** Let \( g \in Y^* \), \( g \neq 0 \). Set \( A_1 = \{ y \in Y ; g(y) > 0 \} \cup \{ 0 \} \) and \( C = \{ y \in Y ; g(y) < 0 \} \cup \{ 0 \} \), and note that \( C \) is convex. Since \( \dim Y \geq 2 \), \( k \) is non-trivial and by Theorem 2 there are \( G_\delta \) sets \( H_\alpha \subset \ker g \), \( \alpha \in [2, \omega_1) \) such that \( H_\alpha \supseteq H_\beta \) for \( 2 \leq \alpha < \beta < \omega_1 \) and \( g \) is non-convex. Let \( A_\alpha = C \cup H_\alpha \) for \( \alpha \in [2, \omega_1) \) and note that \( \bigcup_{\alpha \in [1, \omega_1)} A_\alpha = Y \), but if \( \beta \) is a countable ordinal, then \( \bigcup_{\alpha \in [1, \beta)} A_\alpha \neq Y \). Let \( \{ B_\alpha \subset X \}_{\alpha \in [1, \omega_1)} \) be a uniformly discrete system of balls of radius 1. By Theorem 1 (using \( y_1 = 0 \)) there are mappings \( f_\alpha \in C^k(X; Y) \) such that \( \supp f_\alpha \subset B_\alpha \) and \( f_\alpha(X) = A_\alpha \) for each \( \alpha \in [1, \omega_1) \). It follows that \( f = \sum_{\alpha \in [1, \omega_1)} f_\alpha \in C^k(X; Y) \) and \( f(X) = Y \). On the other hand, if \( Z \subset X \) is a separable subset, then \( Z \) meets at most countably many of the balls \( B_\alpha \) and consequently \( f(Z) \neq Y \).

\( \square \)

To relax the assumption on the existence of a suitable bump in the previous corollary we use the following non-separable variant of [HK] Theorem 4. (We note that in the separable case the assumption is in particular satisfied if there is a non-compact operator from \( X \) into \( \ell_p \), see [HK] Proposition 3.33.)

**Theorem 4.** Let \( X \) be a Banach space for which there is \( T \in \mathcal{L}(X; \ell_p(\Gamma)) \) for some infinite \( \Gamma \) and \( 1 < p < \infty \) such that \( T(B_X) \) contains the canonical basis \( \{ e_\gamma \}_{\gamma \in \Gamma} \). Then for every Banach space \( Y \) of density at most \( \card \Gamma \) there exists a polynomial surjection \( P \in \mathcal{P}(\ell_p(X); Y) \).

**Proof.** Let \( \{ x_\gamma \}_{\gamma \in \Gamma} \subset B_X \) be such that \( T(x_\gamma) = e_\gamma \), \( \gamma \in \Gamma \), and let \( \{ y_\gamma \}_{\gamma \in \Gamma} \) be a dense set in \( B_Y \). Denote \( m = \lfloor p \rfloor \) and define \( Q : \ell_p(\Gamma) \to Y \) by \( Q(z) = \sum_{\gamma \in \Gamma} f_\gamma(z) y_\gamma \), where \( f_\gamma \) are the canonical coordinate functionals. Then \( Q \in \mathcal{P}(\ell_p(\Gamma); Y) \) by [H] Theorem 1.29 (consider the net indexed by the directed set of all finite subsets of \( \Gamma \)). Finally, define \( P \in \mathcal{P}(\ell_p(X); Y) \) by \( P = Q \circ T \). Then if \( y \in Y \), then by [H] Fact 6.64 there is a sequence \( \{ y_\gamma \}_{\gamma \in \Gamma} \) of distinct elements of \( \Gamma \) such that \( y = \| y \| \cdot \sum_{\gamma \in \Gamma} 2^{-m\gamma} y_\gamma \). Put \( x = \| y \| \cdot \sum_{\gamma \in \Gamma} 2^{-m\gamma} x_\gamma \). Then \( P(x) = Q(\| y \| \cdot \sum_{\gamma \in \Gamma} 2^{-m\gamma} e_\gamma) = \| y \| \sum_{\gamma \in \Gamma} 2^{-m\gamma} y_\gamma = y \).

The next corollary in particular solves Problem 186 from [GMZ].

**Corollary 5.** Let \( X \) be a Banach space for which there is \( T \in \mathcal{L}(X; \ell_p(\Gamma)) \) for some uncountable \( \Gamma \) and \( 1 < p < \infty \) such that \( T(B_X) \) contains the canonical basis of \( \ell_p(\Gamma) \). (This holds in particular if \( X \) has a non-separable super-reflexive quotient.) Then for any separable Banach space \( Y \) with \( \dim Y \geq 2 \) there is \( f \in C^\infty(X; Y) \) such that \( f(X) = Y \) but \( f(Z) \neq Y \) for any separable subset \( Z \subset X \).

**Proof.** By Theorem 4 there is a polynomial surjection \( P : X \to \ell_2(\Gamma) \). By Corollary 4 there is \( g \in C^\infty(\ell_2(\Gamma); Y) \) such that \( g(\ell_2(\Gamma)) = Y \) but \( g(Z) \neq Y \) for any separable subset \( Z \subset \ell_2(\Gamma) \). To finish we set \( f = g \circ P \).

If \( X \) is non-separable and super-reflexive, then there is a bounded linear injection from \( X \) into \( \ell_2(\Gamma) \), see e.g. [ITZ] proof of Lemma 2. The existence of the operator \( T \) now follows from Corollary 4[12] used with \( \mu = \omega_1 \). As for the quotient, see the remark preceding Corollary 1[12].

\( \square \)

We note that for \( k > 1 \) the assumption of Corollary 3 is stronger than the assumption of Corollary 5. Indeed, if \( X \) admits a \( C^{k-1} \)-smooth bump, then it is already super-reflexive.

We now proceed to prove Theorem 1. This will be done with the help of the next two auxiliary statements.

**Lemma 6.** Let \( X \) be an infinite-dimensional Banach space that admits a function \( \varphi \in C^k(X; [0, 1]) \), \( k \in \mathbb{N} \cup \{\infty\} \), with each derivative bounded on \( X \), and such that \( \supp \varphi \subset B_X \) and \( \varphi = 1 \) on \( B(0, r) \) for some \( r \in (0, 1) \). Let \( T \) be a rooted \( X \)-branching tree of height \( \omega \). Let \( n_0 \in \mathbb{N} \) and \( \{ n_{n+1} = n \}_{n \in \mathbb{N}} \subset (0, +\infty) \) be such that \( n_0 \to 0 \). Let \( Y \) be a Banach space and let \( \{ y_t \}_{t \in \mathcal{T}} \subset Y \) be such that \( y_0 = 0 \) and \( \| y_t - y_{t-} \| \leq \varepsilon_n ((\frac{t}{r})^k)^n \) for each \( t \in T_{n_0} \), \( n \in \mathbb{N} \), \( n \geq n_0 \) if \( k \in \mathbb{N} \), resp. \( \| y_t - y_{t-} \| \leq \varepsilon_n (\frac{t}{r})^k \) if \( k = \infty \). Then there is \( f \in C^k(X; Y) \) such that \( \supp f \subset B_X \) and \( f(X) = \bigcup_{t \in \mathcal{T} \setminus \{0\}} [y_{t-}, y_t] \cup \{ \lim_{n \to \infty} v_{n,t} ; b \in \mathcal{B}(T) \} \).

**Proof.** Note that \( T_0 = \{0\} \). By induction on the tree levels we find a collection \( \{ x_t \}_{t \in \mathcal{T}} \subset X \) such that

(i) \( U(x_t, (\frac{t}{r})^k) \subset U(x_s, (\frac{s}{r})^k) \) for each \( n \in \mathbb{N} \), \( t \in T_{n-1} \), and \( s \in T_0 \), and

(ii) for each \( n \in \mathbb{N} \) the family \( \{ U(x_t, (\frac{t}{r})^k) \}_{t \in T_n} \) is uniformly discrete.
Set $x_0 = 0$. Let $n \in \mathbb{N}$ and assume that \( \{x_t\}_{t \in T_{n-1}} \) are already defined. By [11] Fact 6.65 each ball $B(x_t, \frac{1}{2}r(\xi^n_{n-1}), t \in T_{n-1}$ contains a $\frac{1}{2}r(\xi^n_{n-1})$-separated set $\{x_t\}_{t \in T_{n-1}}$. Then $U(x_t, \frac{1}{2}r(\xi^n_{n-1}), t \in T_{n-1})$ for each $s \in T^+$ and so (i) holds. Also, each family $\{U(x_t, (\xi^n_{n+1}))\}_{t \in T_{n-1}}$ is $\frac{1}{2}r(\xi^n_{n-1})$-uniformly discrete and combining this with (i) and the inductive hypothesis gives (ii).

Next, for $n \in \mathbb{N}$ and $x \in X$ we set

$$f_n(x) = \sum_{i=1}^n \sum_{t \in T_i} (y_t - y_{t-}) \psi\left(\frac{1}{2^n}((x - x_t))\right).$$

The inner sum is locally finite by (ii) and hence $f_n \in C^K(X; Y)$ and

$$D^j f_n(x) = \sum_{i=1}^n \sum_{t \in T_i} (y_t - y_{t-}) (\frac{1}{2^n})^j D^j \psi\left(\frac{1}{2^n}((x - x_t))\right)$$

for each $x \in X$ and $j < k + 1$. In fact, since $D^j \psi\left(\frac{1}{2^n}((x - x_t))\right)$ is non-zero only for $x \in U(x_t, (\xi^n_{n+1})) \setminus B(x_t, (\xi^n_{n+1}))$, by (i) and (ii) we see that at each $x \in X$ only summand overall in the formula for $D^j f_n(x)$ can be non-zero and so we have the following estimate:

$$\|D^j f_m(x) - D^j f_i(x)\| \leq \max_{i \neq j} \varepsilon_i (\frac{1}{2^n})^j C_j \leq C_j \sup_{i \neq j} \varepsilon_i$$

for $x, m, n \geq l \geq n_0, l \geq j, k < k + 1$, $C_j > 0$ is such that $D^j \psi$ is bounded by $C_j$. It follows by [11] Theorem 1.85 that $f_n \to f \in C^K(X; Y)$ uniformly on $X$.

Finally, note that $\psi(X \setminus U(0, r)) = \psi(B(0, 1) \setminus U(0, r)) = [0, 1]$. Hence, by using induction on $n$ and properties (i) and (ii) we obtain $f_n = B(x_t, r(\xi^n_{n+1})) = y_t$ for each $t \in T_n$ and $f(X \setminus F_n) = f_n(X \setminus F_n) = \bigcup_{i \leq k \leq n} \bigcup_{t \in T_i} [y_t - r, y_t]$.

Recall that the Lindelöf number of a topological space is the smallest infinite cardinal number $\mu$ such that every open covering of this space has a subcovering of cardinality at most $\mu$. For metric spaces the Lindelöf number is equal to the density.

**Proposition 7.** Let $(X, \rho)$ be a metric space with Lindelöf number $\mu, U \subset X$, let $A \subset \bar{U}$ be a non-empty $\mu$-Suslin set, and let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, +\infty)$. Then there is a rooted $\mu$-branching $T$ of height $\omega$ and a family $\{x_t\}_{t \in T} \subset U$ such that $A = \{\lim_{n \to \infty} x_{b_n}; b \in \mathcal{B}(T)\}$ and $\rho(x_u, x_t) < \varepsilon_n$ for each $u \in T^+$, $t \in T_n, n \in \mathbb{N}$.

**Proof.** There is a $\mu$-branching tree $S$ of height $\omega$ and closed sets $F_s \subset U, s \in S$ such that $F_u \subset F_s$ if $u \in s^+$ and $A = \bigcup_{s \in S} \bigcap_{n=1}^{\infty} F_{b_n}$. Without loss of generality we may assume that $S$ is rooted and that $\varepsilon_n \to 0$. We construct the tree $T$ and the family $\{x_t\}_{t \in T}$ by induction on the tree level. The members of $T$ will be pairs $(s, \alpha)$, where $s \in S$. We set the least element of $T$ as $(0, 0)$ and choose $x_{(0,0)} \in U$ arbitrarily. To start the induction we also put $\varepsilon_0 = +\infty$. Let $n \in \mathbb{N}_0$ and assume that $T_n$ and $x_t, t \in T_n$ are already defined. Fix $t = (s, \alpha) \in T_n$. For each $u \in s^+$ there is a covering of $U(x_t, \varepsilon_n) \cap F_u$ by balls $U(x_{(u, \alpha)}, \varepsilon_n + 1)$, $u \in F_{b_n}$ such that $x_{(u, \alpha)} \in U \cap \bigcup_{n \geq l} \bigcup_{t \in T_l} [y_t - r, y_t]$. Fix $t \in T^+$, $t \in T_n$ and clearly $\alpha < \mu$. Then there is a path $\alpha \leq \mu$ and $\rho(x_{(u, \alpha)}, x_t) < \varepsilon_n$ for any $u \in T^+$.

Now for a given $x \in A$ there is a branch $b \in \mathcal{B}(S)$ such that $x \in \bigcap_{n=1}^{\infty} F_{b_n}$. By induction it is easy to see that for each $n \in \mathbb{N}_0$ there is $\beta_n \in T_{n-1}$ such that $x \in F_{b_n} \cap U(x_{(b_n, b_n)}, \varepsilon_n)$. Put $c_0 = (0, 0)$ and $c_n = (b_n, \beta_n)$. Then $\{c_n\}_{n=0}^{\infty}$ is a branch in $T$ and clearly $\lim_{n \to \infty} x_{c_n} = x$. Thus $A = \bigcap_{n=1}^{\infty} x_{c_n} \in \mathcal{B}(T)$. On the other hand, suppose that $c \in \mathcal{B}(T)$ and $x \in \lim_{n \to \infty} x_{c_n}$. Then $c_n = (b_n, \beta_n)$ with $\{b_n\}_{n=0}^{\infty}$ being a branch in $S$. Since $\dist(x_{c_n}, F_{b_n}) < \frac{1}{n}$ and $F_{b_n} \subset F_{b_0}$ for $n \geq k$, it follows that $x \in F_{b_0}$ for each $k \in \mathbb{N}$ and consequently $x \in \bigcap_{n=1}^{\infty} F_{b_n}$. Thus $\{\lim_{n \to \infty} x_{c_n}; d \in \mathcal{B}(T)\} \subset A$.

**Proof of Theorem 7.** By composing the bump with a suitable smooth real function we obtain $r \in (0, 1)$ and $\phi$ as in Lemma 6.

Set $\varepsilon_n = \frac{1}{n} (\frac{1}{n} - \frac{1}{n}^2)$ if $k \in \mathbb{N}_0$, resp. $\varepsilon_n = \frac{1}{n} (\frac{1}{n} - \frac{1}{n}^2)$ if $k = \infty$. By Proposition 7 there is a rooted dens $Y$-branching (and hence also dens $X$-branching) tree $T$ of height $\omega$ and a family $\{y_t\}_{t \in T} \subset C$ such that $A = \{\lim_{n \to \infty} y_{b_n}; b \in \mathcal{B}(T)\}$ and $\|y_t - y_{t-}\| < \varepsilon_{n+1}$ for each $t \in T_n, n \geq 2$. By relabelling and adding a node we may assume that $T_0 = \{0\}$, $T_1 = \{1\}$, and $y_0 = 0$. By Lemma 6 there is $f \in C^K(X; Y)$ such that supp $f \subset B_X$ and $f(X) = A \cup \bigcup_{t \in T_0 \setminus \{0\}} [y_t - y_{t-}] = [0, y_1] \cup A$.

**2. Canonical basis of $\ell_p(\Gamma)$ in a Linear Image**

In this section we look for some sufficient conditions on the space $X$ ensuring that there is a bounded linear operator $T : X \to \ell_p(\Gamma)$ such that $T(B_X)$ contains the canonical basis. For $p = 1$ we have the following simple and certainly well-known observation.
Fact 8. Let $X$ be a Banach space. Then there is $T \in \mathcal{L}(X; \ell_1(\Gamma))$ such that $T(B_X)$ contains the canonical basis $\{e_\gamma\}_{\gamma \in \Gamma}$ if and only if $X$ has a complemented subspace isomorphic to $\ell_1(\Gamma)$.

Proof. $\Leftarrow$ We can take the projection composed with the isomorphism and suitably scaled.

$\Rightarrow$ We can use the lifting property of $\ell_1(\Gamma)$ once we realise that $T$ is in fact onto. Indeed, let $x_\gamma \in B_X$ be such that $T(x_\gamma) = e_\gamma$. Any $y \in \ell_1(\Gamma)$ is of the form $y = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma$. We can put $x = \sum_{\gamma = 1}^\infty a_\gamma x_\gamma$, since the series converges absolutely. Then $T(x) = y$.

\[ \square \]

Now we give several auxiliary technical statements leading finally to the conditions given in Corollary 12.

Let $X$ be a normed linear space and $M \subset X^*$. We say that a net $\{x_\alpha\}_{\alpha \in A} \subset X$ is $M$-null if $\lim_{\alpha} f(x_\alpha) = 0$ for each $f \in M$. Recall that the cofinality of an infinite cardinal $\mu$, denoted by $\text{cf} \mu$, is the smallest cardinal $\nu$ such that $[0, \mu)$ has a subset $A$ of cardinality $\nu$ with $\sup A = \mu$; $\mu$ is called regular if $\text{cf} \mu = \mu$.

Lemma 9. Let $X$ be a normed linear space and $\{f_\gamma\}_{\gamma \in \Gamma} \subset X^*$. For $x \in X$ we denote $\text{supp } x = \{ \gamma \in \Gamma; f_\gamma(x) \neq 0 \}$. Let $\mu > \omega$ be a regular cardinal and $\{x_\alpha\}_{\alpha \in [0, \mu)} \subset X$ an $\{f_\gamma\}$-null net such that $\text{supp } x_\alpha < \mu$ for each $\alpha \in [0, \mu)$. Then there is a subnet $\{x_{\alpha_n}\}_{\alpha_n \in [0, \mu)}$ of $\{x_\alpha\}_{\alpha \in [0, \mu)}$ with disjoint supports, i.e., $\text{supp } x_{\alpha_n} \cap \text{supp } x_{\alpha_m} = \emptyset$ for any $\alpha_n, \alpha_m \in [0, \mu)$, $\alpha_n \neq \alpha_m$.

Proof. Since $\text{cf} \mu > \omega$, for each $\gamma \in \Gamma$ there is $G(\gamma) \in [0, \mu)$ such that $f_\gamma(x_{\alpha_n}) = 0$ for $\alpha_n \in G(\gamma), \mu$. We define an increasing $F : [0, \mu) \to [0, \mu)$ such that $\text{supp } F(x_\alpha) \cap \text{supp } F(x_\beta) = \emptyset$ whenever $0 \leq \alpha < \beta < \mu$ by transfinite recursion. Put $F(0) = 0$. Let $\mu < (0, \mu)$ and put $A = \bigcup_{\alpha \in [0, \mu)} \text{supp } F(x_\alpha)$. Then card $A < \mu$ since $\mu$ is regular. Put $\eta = \sup_{\alpha \in A} G(\gamma)$. Then again $\eta < \mu$ by the regularity. We set $F(\beta) = \max \{ \eta, \text{supp } F(x_\alpha), F(\alpha) + 1 \}$ and note that $f_\gamma(F(x_\alpha)) = 0$ for $\gamma \in \Gamma$.

Finally, we put $y_{\alpha_n} = x_{F(x)}$ for $\alpha_n \in [0, \mu)$, which clearly defines a (Willard) subnet of $\{x_\alpha\}_{\alpha \in [0, \mu)}$.

\[ \square \]

Proposition 10. Let $X$ be a normed linear space and $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some $\Gamma$ and $1 \leq p < \infty$. Let $\mu > \omega$ be a regular cardinal. Then for each weakly null net $\{x_\alpha\}_{\alpha \in [0, \mu)} \subset X \setminus \ker T$ there is a subnet $\{y_{\alpha_n}\}_{\alpha_n \in [0, \mu)}$ in $\mathcal{L}(X; \ell_p(\Gamma))$ such that $S(y_{\alpha_n}) = e_\alpha$, $\alpha \in [0, \mu)$, where $\{e_\alpha\}_{\alpha \in [0, \mu)}$ is the canonical basis of $\ell_p(\{0, \mu\})$.

The same holds if we consider $\text{co}(T)$ and $\text{co}(0, \mu))$ instead of $\ell_p(\Gamma)$ and $\ell_p(\{0, \mu\})$.

Proof. Consider the sets $\Gamma_n = \{ \alpha \in [0, \mu); \frac{1}{n} \leq \|T(x_\alpha)\| \leq n \}$. Since $\text{cf} \mu > \omega$, there is $n \in \mathbb{N}$ such that card $\Gamma_n = \mu$. Thus by passing to a subnet we may assume that $\{T(x_\alpha)\}_{\alpha \in [0, \mu)}$ is semi-normalised. Since $T$ is $w^*-w$ continuous, $\{T(x_\alpha)\}_{\alpha \in [0, \mu)}$ is weakly null. By Lemma 9 there is a subnet $\{y_{\alpha_n}\}_{\alpha_n \in [0, \mu)}$ of $\{x_\alpha\}_{\alpha \in [0, \mu)}$ such that $\{T(y_{\alpha_n})\}_{\alpha_n \in [0, \mu)}$ have disjoint supports. Consequently there is a bounded linear projection $P : \ell_p(\Gamma) \to \text{span}\{T(x_\alpha)\}$ and an isomorphism $R \in \mathcal{L}(\text{span}\{T(y_{\alpha_n})\}; \ell_p(\{0, \mu\}))$ with $R(T(y_{\alpha_n})) = e_\alpha$. We may then set $S = R \circ P \circ T$.

\[ \square \]

Let $X$ be an infinite-dimensional normed linear space and $\mu$ a cardinal. For an application of Proposition 10 we need to find a non-trivial weakly null long sequence in $X$. Notice that if $M \subset X^*$ separates the points of $X$, then there is no $\sigma(X, M)$-null long sequence of length $\mu$ in $X \setminus \ker M$. Indeed, $M$ gives rise to a neighbourhood basis of $\sigma(X, M)$ of cardinality $M$ and thus any $\sigma(X, M)$-null long sequence of length $\mu$ is eventually zero. Consequently, there is no weakly null long sequence of length $\mu$ in $X \setminus \ker M$ if $\text{cf} \mu > w^*$-dens $X$ and there is no non-zero $w^*$-null long sequence of length $\omega_1$ in $\ell_{c_0}$. In particular, if we want to have a weakly null long sequence of length $\sup X \setminus \ka 0 \setminus \sup X \setminus \ka 0 \setminus X$ a regular cardinal, then the space $X$ has to be a dense subspace, i.e., a space for which $w^*$-dens $X = \sup X$.

Recall that a Banach space $X$ is weakly Lindelöf determined (WLD) if and only if there is a one-to-one $w^*$-pointwise continuous bounded linear operator $T : X^* \to \ell_\infty(\Gamma)$ for some set $\Gamma$, see [AM]. Clearly, a quotient of a WLD space is again WLD. Note also that a WLD space, i.e., a space for which $w^*$-dens $X = \sup X$.

Recall that a Banach space $X$ is weakly Lindelöf determined (WLD) if and only if there is a one-to-one $w^*$-pointwise continuous bounded linear operator $T : X^* \to \ell_\infty(\Gamma)$ for some set $\Gamma$, see [AM]. Clearly, a quotient of a WLD space is again WLD. Note also that a WLD space, i.e., a space for which $w^*$-dens $X = \sup X$.

**Corollary 12.** Let $X$ be a Banach space, $\mu > \omega$ a regular cardinal, $\Gamma$ a set, and $1 \leq p < \infty$. Consider the following conditions:

(i) $X$ is WLD and there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ such that $\text{dens } X / \ker T \geq \mu$.

(ii) $X$ contains a non-zero weakly null net $\{x_\alpha\}_{\alpha \in [0, \mu)}$ and there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ such that $\text{dens } T \geq \mu$.
If one of the above conditions is satisfied, then there is \( S \in \mathcal{L}(X; \ell_p([0, \mu])) \) such that \( S(B_X) \) contains the canonical basis of \( \ell_p([0, \mu]) \).

**Proof.** (i) Let \( X = \mathbb{Q} / \ker T \). According to the remark preceding Corollary 12, it suffices to find the required operator from \( Z \). Let \( Q : X \to Z \) be the canonical quotient mapping. Define \( \tilde{T} : Z \to \ell_p(T) \) by \( \tilde{T}(z) = T(x) \) for some \( x \in Q^{-1}(z) \). Then \( \tilde{T} \in \mathcal{L}(Z; \ell_p(T)) \) and it is one-to-one. Also, \( Z \) is WLD with dens \( Z \geq \mu \) and hence by combining Lemma 11 and Proposition 10 there exists \( S \in \mathcal{L}(Z; \ell_p([0, \mu])) \) such that \( S(B_Z) \) contains the canonical basis of \( \ell_p([0, \mu]) \).

(ii) Since \( cf \mu > \omega \), we may assume without loss of generality that \( \{x_\alpha\}_{\alpha \in [0, \mu]} \) is semi-normalised and contained in \( B_X \). Also, by [11, Theorem 1.1] we may assume that \( \{x_\alpha\}_{\alpha \in [0, \mu]} \) is a long Schauder basic sequence, and in particular that it is uniformly separated. Consequently there is \( \beta < \mu \) such that \( x_\alpha \notin \ker T \) for \( \alpha \geq \beta \) and we may apply Proposition 10.

\[ \square \]

Finally, note that \( \ell_\infty \) has a quotient isomorphic to \( \ell_2(\omega) \) ([HMVZ, Theorem 4.22]) although it does not contain a non-zero \( w^*\)-null long sequence of length \( \omega_1 \).

**Acknowledgement**

We would like to thank Petr Holický for showing Hausdorff’s theorem to us.

**References**


