ON C^1 WHITNEY EXTENSION THEOREM IN BANACH SPACES

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ABSTRACT. Our note is a complement to recent articles [JS1] (2011) and [JS2] (2013) by M. Jiménez-Sevilla and L. Sánchez-González which generalise (the basic statement of) the classical Whitney extension theorem for C^1-smooth real functions on $\mathbb{R}^n$ to the case of real functions on X ([JS1]) and to the case of mappings from X to Y ([JS2]) for some Banach spaces X and Y. Since the proof from [JS2] contains a serious flaw, we supply a different more transparent detailed proof under (probably) slightly stronger assumptions on X and Y. Our proof gives also extensions results from special sets (e.g. Lipschitz submanifolds or closed convex bodies) under substantially weaker assumptions on X and Y. Further, we observe that the mapping $F \in C^1(X;Y)$ which extends f given on a closed set $A \subset X$ can be, in some cases, C^∞-smooth (or C^k-smooth with $k > 1$) on $X \setminus A$. Of course, also this improved result is weaker than Whitney's result (for $X = \mathbb{R}^n$, $Y = \mathbb{R}$) which asserts that F is even analytic on $X \setminus A$. Further, following another Whitney's article and using the above results, we prove results on extensions of C^1-smooth mappings from open ("weakly") quasiconvex subsets of X. Following the above mentioned articles [JS1], [JS2] we also consider the question concerning the Lipschitz constant of F if f is a Lipschitz mapping.

1. INTRODUCTION

Following articles [JS1] and [JS2] by Mar Jiménez-Sevilla and Luis Sánchez-González we investigate the validity of versions of C^1 Whitney extension theorem for mappings between Banach spaces. The celebrated Whitney extension theorem of [Wh1] gives a condition which is necessary and sufficient for a function $f: A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ is closed, to be extendable to a function $F \in C^k(\mathbb{R}^n)$, $k \in \mathbb{N} \cup \{\infty\}$. For $k = 1$ this condition (which is a special case of condition ($\mathrm{W_{or}}$) from Subsection 2.4) can be easily reformulated (see Subsection 2.4 for a detailed explanation) to the following condition:

(W) There exists a continuous mapping $G: A \to (\mathbb{R}^n)^*$ such that $G(x)$ is a strict derivative of f at x with respect to A for each $x \in A$.

Using this notation, the Whitney C^1 extension theorem ([Wh1, Theorem I] for $k = 1$) can be reformulated as follows:

Theorem W. *If $A \subset \mathbb{R}^n$ is closed and $f: A \to \mathbb{R}$ satisfies condition (W) with $G: A \to (\mathbb{R}^n)^*$, then there is $F \in C^1(\mathbb{R}^n)$ such that*

(a) $F\restriction_A = f$ and
(b) $DF(x) = G(x)$ for each $x \in A$.

Moreover, [Wh1, Theorem I] asserts additionally also the "exterior regularity" of F:

(ER) F is analytic on $\mathbb{R}^n \setminus A$.

However, nowadays it is not generally considered an integral part of Whitney's extension theorem.

The proof of [Wh1, Theorem I] is substantially finite-dimensional and John Campbell Wells in [We] showed that Whitney's theorem does not hold for functions from $C^3(\ell_2)$. (In fact, it does not hold for functions from $C^3(X)$, where X is any infinite-dimensional Banach space; this is proved in [J2].) However, the C^1 case is different, since the important recent article [JS1] contains a generalisation of Theorem W (without statement (b)) in certain infinite-dimensional Banach spaces. Note that this interesting result was not quite surprising, since some Whitney-type extension theorems for $C^{1,1}$-smooth functions (i.e., functions whose derivative is Lipschitz) in some infinite-dimensional spaces were proved in [We] (1973) and [G] (2009). For references to other articles on this topic see [AM], where some extension theorems for (more general) $C^{1,\omega}$-smooth functions on some super-reflexive Banach spaces are contained (for an alternative treatment see [JKZ], where the classical Whitney-Glaeser condition for $C^{1,\omega}$-smooth case is used).

It is interesting that Whitney's proof of Theorem W, known proofs for the $C^{1,1}$-smooth case in infinite dimensional spaces, and the proof of [JS1] are mutually quite different. In particular, [Wh1] and [JS1] use quite different partitions of unity and proofs for the $C^{1,1}$-smooth case do not use any partition of unity at all.

Note that [JS1] generalises results of [AFK1] (with [AFK2]), where extensions of C^1-smooth functions from closed linear subspaces are considered. Roughly speaking, the proofs of [AFK1] and [JS1] have the following main ingredients:

(a) Lipschitz extendability of Lipschitz functions,
(b) approximability of Lipschitz functions by C^1-smooth Lipschitz functions,
(c) the existence of certain C^1-smooth Lipschitz partition of unity,
(d) defining the extension as the limit of successive C^1-smooth approximations.

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The original aim of our research was to write a short remark which using the results of [JS2] shows that, first, for some spaces X, Y we can assert that the extension is C^∞-smooth on the complement of A and, second, that for some X, Y there exists a simple natural necessary and sufficient condition for extendability of C^1-smooth mappings defined on open quasiconvex sets. However, then we observed that the proof of the main theorems [JS2, Theorems 3.1 and 3.2] (which assume that the pair (X, Y) has property called (*)) contains an essential flaw (see Subsection 2.5 for details). Subsequently, we realised that combining some methods from [HJ1] and [JS1] we obtain relatively short and transparent proof of the following result (which is a simplified version of Theorems 34 and 35).

Theorem 1. *Let X, Y be Banach spaces such that the pair (X, Y) has properties (LE) and (LA$_1$). Let $A \subset X$ be closed and suppose that $f\colon A \to Y$ satisfies condition (W) with a mapping $G\colon A \to \mathcal{L}(X; Y)$ (see Definition 25). Then f can be extended to a mapping $g \in C^1(X; Y)$.*

Moreover, if f is L-Lipschitz and $\sup_{x\in A} \|G(x)\| \le L$, then we can additionally assert that g is KL-Lipschitz, where the constant $K > 0$ depends on X and Y only.

Note that properties (LE) (denoted by (E) in [JS2]), see Definition 8, and (LA$_1$) (equivalent to (A) from [JS2], see Remark 17) together imply condition (*) and so Theorem 1 would follow from [JS2, Theorems 3.1 and 3.2]. We do not know whether [JS2, Theorems 3.1 and 3.2] hold, however the proofs in [JS2] can probably be changed to correctly prove Theorem 1 (and so this vector-valued version is essentially due to the authors of [JS2]).

In any case, we believe that our proof of Theorem 1 (resp. Theorems 34, 35) is worth publishing, since it is substantially more detailed and transparent than proofs from [JS2] (and [JS1]).

Moreover, our proof (which works with property (LLE) which is weaker than property (LE)) gives extension results from special sets (e.g. Lipschitz submanifolds and closed convex bodies) under substantially weaker assumptions on X and Y (see Corollary 44 and Remark 45).

We also prove explicitly the generalisation of condition (b) from Theorem W, which is contained only implicity in [JS1] and [JS2]. (Let us note that none of the articles [JS1] or [JS2] refers to the seminal article [Wh1].)

Our further contributions, which are new also in the case of real functions, are the following:

In Section 4 we observe that the mapping $g \in C^1(X; Y)$ which extends the mapping f given on a closed set $A \subset X$ can be, in some cases, C^∞-smooth (or C^k-smooth with $k > 1$) on $X \setminus A$. Of course, also this improved result is weaker than Whitney's result (ER) which asserts that g is even analytic on $X \setminus A$ (for $X = \mathbb{R}^n$, $Y = \mathbb{R}$).

In Section 5, following another Whitney's article [Wh2] and using the above results, we prove results on extensions of C^1-smooth mappings from open ("weakly") quasiconvex subsets of X.

Finally we note that we give a review (much more complete and detailed than the information in [JS1], [JS2]) concerning the pairs (X, Y) for which conditions (LE) and (LA$_1$) hold, see Examples 9, 21, and 43 (and Subsection 2.6).

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After the submission of our paper the preprint [JiS] appeared. It explains how to overcome the flaw in [JS2] with the additional assumption (E) (called (LE) above). In particular, the proof of Theorem 1 above using the methods of [JS2] is described. Surprisingly, the explicit constant $K = K_{X,Y}$ in Theorem 1 obtained in [JiS] is much better than in [JS1], [JS2], and also better than in Section 6 of [JZ] (and is possibly even the best possible one). This nice new achievement of [JiS] led us to the interesting observation that the second (Lipschitz) part of Theorem 1 with "almost the same" constant $K = K_{X,Y}$ as in [JiS] can be rather easily deduced from the first (C^1) part of Theorem 1 via two results from [HJ] (see the proof of Theorem 35 and Remark 37). Accordingly, Theorems 35, 42, and 52 in the present version are proved with better Lipschitz constants than the corresponding results in [JZ], and the whole Section 6 from [JZ] was removed.

2. Preliminaries

2.1. **Basic notation.** By $U(x, r)$ we denote the open ball in a metric space centred at x with radius $r > 0$. An L-Lipschitz mapping is a mapping with a (not necessarily minimal) Lipschitz constant L.

All the normed linear spaces considered are real. Let X, Y be normed linear spaces. By U_X, resp. B_X, resp. S_X we denote the open unit ball of X, resp. the closed unit ball of X, resp. the unit sphere of X. For $x \in X$ and $g \in X^*$ we will denote the evaluation of g at x also by $g[x]$. By $Df(x)$ we will denote the Fréchet derivative of $f\colon A \to Y$, $A \subset X$, at $x \in A$, its evaluation in $h \in X$ will be denoted by $Df(x)[h]$. By $\mathcal{L}(X; Y)$ we denote the space of continuous linear operators from X to Y. By $C^k(\Omega; Y)$ we denote the vector space of C^k-smooth mappings from an open subset $\Omega \subset X$ to Y, as usual we shorten $C^k(\Omega) = C^k(\Omega; \mathbb{R})$.

For a mapping $f\colon X \to Y$, where X is a set and Y is a vector space, we denote $\operatorname{supp_o} f = f^{-1}(Y \setminus \{0\})$.

Recall that a system $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of functions on a set X is called a partition of unity if

- $\psi_\alpha\colon X \to [0, 1]$ for all $\alpha \in \Lambda$,
- $\sum_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for each $x \in X$.

We say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is subordinated to a covering $\mathcal{U}$ of X if $\{\operatorname{supp_o} \psi_\alpha\}_{\alpha\in\Lambda}$ refines $\mathcal{U}$, i.e. for each $\alpha \in \Lambda$ there is $U \in \mathcal{U}$ such that $\operatorname{supp_o} \psi_\alpha \subset U$. Further, in case that X is a topological space we say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite if the system $\{\operatorname{supp_o} \psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite, i.e. if for each point $x \in X$ there is a neighbourhood U of x such that the set $\{\alpha \in \Lambda;\ \operatorname{supp_o} \psi_\alpha \cap U \neq \emptyset\}$ is finite; we say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is σ-discrete if the system $\{\operatorname{supp_o} \psi_\alpha\}_{\alpha\in\Lambda}$

is σ-discrete, i.e. if we can write $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ so that each system $\{\operatorname{supp}_o \psi_\alpha\}_{\alpha \in \Lambda_n}$ is discrete, i.e. for each point $x \in X$ there is a neighbourhood U of x such that $\operatorname{supp}_o \psi_\alpha \cap U \neq \emptyset$ for at most one $\alpha \in \Lambda_n$.

2.2. **Lipschitz retracts, Lipschitz domains, and Lipschitz submanifolds.** Recall that a retraction of a set X onto its subset $Y \subset X$ is a mapping $r\colon X \to Y$ such that $r\restriction_Y = \mathrm{id}_Y$. If such a retraction exists, we say that Y is a retract of X. If X is a metric space and there is a Lipschitz (resp. L-Lipschitz) retraction of X onto $Y \subset X$, then we say that Y is a Lipschitz (resp. L-Lipschitz) retract of X.

Definition 2. A metric space is called an absolute Lipschitz retract if it is a Lipschitz retract of every metric space containing it as a subspace.

Note that each absolute Lipschitz retract is complete: It is a Lipschitz retract of its completion, but clearly each continuous retract of a Hausdorff space X is closed in X.

We will need the following obvious facts.

Fact 3. *Let X_1, X_2 be metric spaces, let Y_1 be a Lipschitz retract of X_1, and let $\Phi\colon X_1 \to X_2$ be a bi-Lipschitz bijection. Then $\Phi(Y_1)$ is a Lipschitz retract of X_2.*

Proof. The retraction can be given by $\Phi \circ r \circ \Phi^{-1}$, where $r\colon X_1 \to Y_1$ is a Lipschitz retraction onto Y_1. □

Fact 4. *Let X, Y be metric spaces and let Z be a K-Lipschitz retract of X. Then each L-Lipschitz mapping $f\colon Z \to Y$ can be extended to a KL-Lipschitz mapping $\tilde{f}\colon X \to Y$.*

Proof. Put $\tilde{f} = f \circ r$, where $r\colon X \to Z$ is a K-Lipschitz retraction onto Z. □

Lemma 5. *Let X be a normed linear space and let $A \subset X$ be an image of a closed convex bounded set with a non-empty interior under a bi-Lipschitz automorphism of X. Then A is a Lipschitz retract of X.*

Proof. A closed convex bounded set with a non-empty interior is a Lipschitz retract of X by [F] and so A is also a Lipschitz retract of X by Fact 3. □

Following essentially [P, Definitions 2, 3] we define the following rather general notion of a Lipschitz submanifold of a normed linear space X:

Definition 6. We say that a non-empty subset A of a normed linear space X is a Lipschitz submanifold of X if for each $x \in A$ there exist an open neighbourhood U of x, two non-trivial normed linear spaces E_1, E_2, and a bi-Lipschitz mapping Φ of U onto $U_{E_1} \times U_{E_2} \subset E_1 \oplus_\infty E_2$ such that $\Phi(A \cap U) = U_{E_1} \times \{0\}$.

Obviously, each complemented proper linear subspace of X is a Lipschitz submanifold of X.

A natural generalisation of a "weakly Lipschitz domain" in $\mathbb{R}^n$ (cf. e.g. [GMM]) to normed linear spaces is the following:

Definition 7. We say that an open non-empty subset G of a normed linear space X is a Lipschitz domain in X if for each $a \in \partial G$ there exist an open neighbourhood V of a, a normed linear space E, and a bi-Lipschitz mapping Φ of V onto $U_E \times (-1, 1) \subset E \oplus_\infty \mathbb{R}$ such that $\Phi(G \cap V) = U_E \times (0, 1)$ and $\Phi(\partial G \cap V) = U_E \times \{0\}$.

Obviously, if $G \neq X$ is a Lipschitz domain in X, then ∂G is a Lipschitz submanifold of X (of "codimension 1").

2.3. **Lipschitz extension and approximation.** In this subsection we discuss facts related to the "ingredients" (a)–(c) mentioned in Introduction. (Variants of some of these facts are presented already in [JS2].)

2.3.1. *Lipschitz extension.* The following Lipschitz extension property is an important notion which was studied in a number of articles.

Definition 8. Let X, Y be normed linear spaces. We say that the pair (X, Y) has property (LE) if there is $C > 0$ such that for every $A \subset X$ every L-Lipschitz mapping $f\colon A \to Y$ has a CL-Lipschitz extension to the whole of X. In this case we say that the pair (X, Y) has property (LE) with C.

Recall that in [JS2] this property is called (E). Some information about pairs with property (LE) are gathered in the following example.

Example 9. The classical result of Hassler Whitney [Wh1, p. 63] and Edward James McShane [MS] on extension of Lipschitz functions (cf. [HJ, Lemma 7.39]) implies that the pair $(X, \mathbb{R})$ has property (LE) for any normed linear space X. The following pairs (X, Y) are known to possess property (LE):

- X is any normed linear space, Y is an absolute Lipschitz retract, see [BL, Proposition 1.2 and the Remark (iii) after]. The following spaces are known to be absolute Lipschitz retracts:
  (a) $\ell_\infty(\Gamma)$, see [HJ, Fact 7.76]; in particular finite-dimensional spaces.

(b) $B_0(V)$, the space of all bounded real-valued functions f on a topological space V with a distinguished point $v_0 \in V$ for which $f(v) \to 0$ whenever $v \to v_0$, considered with the supremum norm. This is a result of Joram Lindenstrauss, see [Li, Theorem 6]. It follows that $c_0(\Gamma)$ is an absolute Lipschitz retract.

(c) $C_{\mathrm{ub}}(P)$, the space of all bounded uniformly continuous real-valued functions on a metric space P with the supremum norm. In particular $C(K)$, where K is a metric compact space. This is a result of J. Lindenstrauss, see [BL, Theorem 1.6].

- X is finite-dimensional, Y is an arbitrary Banach space, see [JLS, Theorem 2]. The method of the proof is in fact the same as that of the smooth Whitney extension theorem from [Wh1].
- X is a normed linear space that has an equivalent norm with modulus of smoothness of power type 2, Y is a Banach space that has an equivalent norm with modulus of convexity of power type 2. This goes back to the classical theorem of Mojżesz D. Kirszbraun (for X, Y Hilbert spaces) and it is a combination of results of Keith Ball ([B, Theorem 1.7, Theorem 4.1]) and Assaf Naor, Yuval Peres, Oded Schramm, and Scott Sheffield ([NPSS, Theorem 1.2]).

On the other hand, there are pairs of spaces that do not have property (LE), for example (ℓ_p, ℓ_q) for $1 \leq p < q < \infty$. This follows from [N, the proof on p. 266–268] if we replace the space ℓ_2 with ℓ_q and take the corresponding Mazur mappings $\varphi\colon \ell_p \to \ell_q$, resp. $\varphi_n\colon \ell_p^{2n} \to \ell_q^{2n}$, cf. the remark in the proof of [N, Proposition 5]. Consequently, the pair (X, Y) does not have property (LE) whenever X contains a subspace isomorphic to ℓ_p and Y contains a complemented subspace isomorphic to ℓ_q for some $1 \leq p < q < \infty$. In particular, the pair $(L_p(\mu), L_q(\nu))$ does not have property (LE) if $1 \leq p < q < \infty$ for any measures μ, ν such that both $L_p(\mu)$ and $L_q(\nu)$ are infinite-dimensional (use [AK, Proposition 6.4.1] and its proof).

We remark that in the context of L_p-spaces it is an open question whether the pair $\big(L_2([0,1]), L_1([0,1])\big)$ has property (LE), see [NPSS].

We will prove our extension results using the following localised version of the Lipschitz extension property.

Definition 10. Let X, Y be normed linear spaces and $A \subset X$. We say that the pair (A, Y) has property (LLE) if for every $x \in A$ there is $K \geq 1$ such that for each $\delta > 0$ there is an open neighbourhood U of x such that $U \subset U(x, \delta)$ and each Q-Lipschitz mapping from $U \cap A$ to Y can be extended to a KQ-Lipschitz mapping on U.

Remark 11. It is clear that if the pair (X, Y) of normed linear spaces has property (LE), then for every $A \subset X$ the pair (A, Y) has property (LLE).

Remark 12. If the pair (A, Y) has property (LLE) and $B \subset A$ is relatively open in A, then the pair (B, Y) also has property (LLE).

Indeed, fix $x \in B$. Let $K \geq 1$ be the constant from property (LLE) of (A, Y) for this x. Let $\delta > 0$ be given and let $0 < \eta \leq \delta$ be such that $U(x, \eta) \cap A \subset B$. By the property (LLE) of (A, Y) there is an open neighbourhood U of x in X such that $U \subset U(x, \eta) \subset U(x, \delta)$ and each Q-Lipschitz mapping from $U \cap B = U \cap A$ to Y can be extended to a KQ-Lipschitz mapping on U.

Lemma 13. *Let X be a normed linear space and suppose that $A \subset X$ has the following property:*

($\widetilde{\mathrm{R}}$) *For each $x \in A$ there exists an open neighbourhood W of x and a Lipschitz retraction from W onto $A \cap W$.*

Then for each normed linear space Y the pair (A, Y) has property (LLE).

Proof. Let $x \in A$ and let W be an open neighbourhood of x and $r\colon W \to A \cap W$ a Lipschitz retraction. Choose $K \geq 1$ such that r is K-Lipschitz. Let $\delta > 0$ be given. Choose $0 < \eta \leq \delta$ such that $U(x, \eta) \subset W$ and set $U = U(x, \eta) \cap r^{-1}(A \cap U(x, \eta))$. Since $A \cap U(x, \eta)$ is an open subset of $A \cap W$, we obtain that $U \subset U(x, \eta)$ is an open neighbourhood of x and $r\restriction_U$ is a K-Lipschitz retraction of U onto $A \cap U(x, \eta) = A \cap U$. So each Q-Lipschitz mapping f from $U \cap A$ to Y can be extended to a KQ-Lipschitz mapping $f \circ r\restriction_U$ on U.

□

Remark 14. Note that each Lipschitz retract of an open subset of a normed linear space has property ($\widetilde{\mathrm{R}}$).

Lemma 15. *Let X, Y be normed linear spaces and suppose that $A \subset X$ is one of the following types:*

(a) A is an image of a closed convex bounded set with a non-empty interior under a bi-Lipschitz automorphism of X;

(b) A is a Lipschitz submanifold of X;

(c) A is the closure of a Lipschitz domain in X.

Then the pair (A, Y) has property (LLE).

Proof. By Lemma 13 it is sufficient to prove that A has property ($\widetilde{\mathrm{R}}$).

In the case (a) we use Lemma 5 together with Remark 14.

In the case (b), let an arbitrary $x \in A$ be fixed and let U, E_1, E_2, and Φ be as in Definition 6. Since $U_{E_1} \times \{0\}$ is clearly a Lipschitz retract of $U_{E_1} \times U_{E_2}$, Fact 3 implies that $A \cap U$ is a Lipschitz retract of U. So we have proved that A has property ($\widetilde{\mathrm{R}}$).

In the case (c), let $G \subset X$ be a Lipschitz domain in X such that $A = \overline{G}$. Consider an arbitrary point $a \in \partial G$ and choose V, E, and Φ as in Definition 7 (with $a := x$). Then $\Phi(A \cap V) = U_E \times [0, 1)$. Since $U_E \times [0, 1)$ is clearly a Lipschitz retract of $U_E \times (-1, 1)$, Fact 3 implies that $A \cap V$ is a Lipschitz retract of V. Since the case $a \in G$ is trivial, we have proved that A has property ($\widetilde{\mathrm{R}}$).

□

2.3.2. *Smooth approximation of Lipschitz mappings.* The approach used already in [AFK1] is based on the smooth approximation of Lipschitz mappings; we introduce the following terminology:

Definition 16. Let X, Y be normed linear spaces. We say that the pair (X, Y) has property (LA_k), $k \in \mathbb{N} \cup \{\infty\}$, if there is $C \geq 0$ such that for any L-Lipschitz mapping $f : U_X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X; Y)$ such that $\sup_{U_X} \|f - g\| \leq \varepsilon$. In this case we say that the pair (X, Y) has property (LA_k) with C. We say that X has property (LA_k) (resp. (LA_k) with C) if the pair $(X, \mathbb{R})$ has this property.

Clearly, if $k > l$ and the pair (X, Y) has property (LA_k), then it also has property (LA_l).

Remark 17. Property (LA_1) is equivalent to property (A) of [JS2], i.e. the ability to approximate mappings on the whole space (Definition 16 with $k = 1$ and U_X replaced by X). Indeed, $\Rightarrow$ follows from [HJ, Theorem 7.86]. $\Leftarrow$ follows from [HJ, Theorem 7.86] again – it suffices to show that (A) implies the assumption of [HJ, Theorem 7.86]: given a Lipschitz $h : 2U_X \to Y$, using the fact that B_X is a 2-Lipschitz retract of X (see e.g. [F, Section 2]) we are able to extend $h\restriction_{B_X}$ to the whole X and we can use property (A) to approximate h on U_X. (Note that the use of [HJ, Theorem 7.86] is necessary: the space Y may not be complete, so we may not be able to extend f from U_X to B_X to use the retraction to B_X.)

In a similar vein, it is easy to observe that property (LA_k) is equivalent to the ability to approximate mappings on the whole space (Definition 16 with U_X replaced by X).

Remark 18. It may seem perhaps more natural to use the formulation as in (A) of [JS2] instead of (LA_1). However, our proofs require precisely the approximation on balls so for us it is actually more natural to use the formulation as in (LA_1). The relation of the constants of properties (A) and (LA_1) is as follows: If the pair (X, Y) has property (LA_1) with constant C, then for any $\eta > 1$ it has property (A) with constant ηC. This follows from [HJ, Theorem 7.86]. On the other hand if (X, Y) has property (A) with constant C, then for any $\eta > 1$ it has property (LA_1) with constant $\eta 2C$. This is explained in Remark 17. However, if (X, Y) has property (LE) with constant 1, then property (A) with constant C implies property (LA_1) also with constant C. This is the case for example if $Y = \mathbb{R}$, $Y = \ell_\infty(\Gamma)$, or X and Y are Hilbert spaces. In general we do not know whether property (A) with constant C implies (LA_1) with a better constant than $2C$.

Remark 19. It is easy to see that if some pair (X, Y), Y non-trivial, has property (LA_k), then X also has (LA_k) with the same constant. Indeed, if $f : U_X \to \mathbb{R}$ is L-Lipschitz and $\varepsilon > 0$, then choose some $y \in S_Y$ and consider the mapping $\bar{f} : U_X \to Y$, $\bar{f}(x) = f(x) \cdot y$. Let $\bar{g} \in C^k(U_X; Y)$ be a CL-Lipschitz ε-approximation of $\bar{f}$ provided by the property (LA_k) of the pair (X, Y). Let $F \in Y^*$ be a Hahn-Banach extension of the norm-one functional $ty \mapsto t$ defined on $\operatorname{span}\{y\}$. Then $g = F \circ \bar{g} \in C^k(U_X)$ is the desired CL-Lipschitz ε-approximation of the function f.

Remark 20. Note that property (LA_k) with constant C easily implies (via translation and scaling of the domain) that for any $x \in X$, $r > 0$, any L-Lipschitz mapping $f : U(x, r) \to Y$, and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U(x, r); Y)$ such that $\sup_{U(x,r)} \|f - g\| \leq \varepsilon$.

Example 21. The following pairs (X, Y) are known to possess property (LA_k):

(a) X is such that there are a set Γ and a bi-Lipschitz homeomorphism $\Phi : X \to c_0(\Gamma)$ into such that the component functions $e^*_\gamma \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$, Y is a Banach space, and X or Y is an absolute Lipschitz retract. See [HJ, Theorem 7.79] with Remark 17. Particular examples of such pairs are:
  - (a1) X is finite-dimensional, Y is an arbitrary Banach space, and $k = \infty$.
  - (a2) $X = c_0(\Gamma)$, Y is an arbitrary Banach space, and $k = \infty$.
  - (a3) X is separable and admits a C^k-smooth Lipschitz bump, Y is a Banach space, and X or Y is an absolute Lipschitz retract. See [HJ, Corollary 7.65], cf. [HJ, Corollary 7.81].
  - (a4) X is a subspace of $L_p(\mu)$ for some measure μ and $1 < p < \infty$, resp. of some super-reflexive Banach lattice with a (long) unconditional basis or a weak unit, with $\operatorname{dens} X < \omega_\omega$, Y is a Banach space that is an absolute Lipschitz retract, and $k = 1$. See [HJS, Corollary 29].

(b) X is a Banach space with an unconditional Schauder basis that admits a C^k-smooth Lipschitz bump, Y is an arbitrary Banach space. See [HJ, Corollary 7.87].

(c) X is a super-reflexive space, Y is finite-dimensional, and $k = 1$. See [J1] combined with Remarks 17 and 28.

In particular, note that if X^* is separable, then by [DGZ, Theorem II.3.1], (a3) above, and Example 9(a) the space X has property (LA_1).

2.3.3. *Partitions of unity.* Smooth approximations theorems are tightly connected with the existence of smooth partitions of unity. The following lemma is a weaker version of [HJ, Lemma 7.85].

Lemma 22. *Let X be a normed linear space with property (LA_1) and let $\Omega \subset X$ be open. Then for any open covering $\mathcal{U}$ of Ω there is a Lipschitz and C^1-smooth locally finite and σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$.*

Proof. The proof of [HJ, Lemma 7.85] lacks details, so we give a more elaborated argument. We will use [HJ, Lemma 7.49]. Let $\emptyset \neq G \subset X$ be open and put $S(G) = \{f \in C^1(G);\ f \text{ is bounded and Lipschitz}\}$. Then $S(G)$ is a partition ring (see [HJ, Definition 7.47]). Indeed, it is clearly a ring. To show property (i) of a partition ring, let $\{f_\gamma\}_{\gamma \in \Lambda} \subset S(G)$ be such that $\{\operatorname{supp}_\circ f_\gamma\}_{\gamma \in \Lambda}$ is uniformly discrete. For each $\gamma \in \Lambda$ let $g_\gamma = c_\gamma f^2_\gamma$ for some suitable constant $c_\gamma > 0$ chosen so that g_γ is

1-Lipschitz and bounded by 1. Put $g = \sup_{\gamma \in \Lambda} g_\gamma$. Obviously g is bounded and Lipschitz and it is easily seen that $g \in C^1(G)$. So $g \in S(G)$ and clearly $\operatorname{supp}_o g = \bigcup_{\gamma \in \Lambda} \operatorname{supp}_o f_\gamma$.

Property (ii): Let $f \in S(G)$ and $\operatorname{supp}_o f = U_1 \cup U_2$, where U_1 and U_2 are open subsets of G with $d = \operatorname{dist}(U_1, U_2) > 0$. Let $L, M \geq 0$ be such that f is L-Lipschitz and $|f(x)| \leq M$ for each $x \in G$. Consider the function $g = \chi_{U_1} f$. Then $g = f$ on the open set $G \setminus \overline{U_2}$ and $g = 0$ on some neighbourhood of $\overline{U_2}$, hence $g \in C^1(G)$. To see that g is Lipschitz, observe that if $x \in U_1$ and $y \in U_2$, then $|g(x) - g(y)| = |f(x)| \leq M \leq \frac{M}{d}\|x - y\|$. By inspecting all other (easy) cases we obtain that g is $\max\{L, \frac{M}{d}\}$-Lipschitz and so $g \in S(G)$.

Property (iii): Let $f \in S(G)$ and $\varepsilon > 0$. Let $\psi \in C^1(\mathbb{R})$ be such that $0 \leq \psi \leq 1$, $\psi(t) = 0$ for $t \leq \varepsilon$, and $\psi(t) = 1$ for $t \geq 2\varepsilon$. Put $g = \psi \circ f$. Since ψ is Lipschitz, it follows that $g \in S(G)$ and it clearly has the properties required in (iii).

Now to show that (ii) of [HJ, Lemma 7.49] for $S = S(\Omega)$ is satisfied let $V \subset W \subset \Omega$ be bounded open sets satisfying $\delta = \operatorname{dist}(V, \Omega \setminus W) > 0$. If $W = \Omega$, then we set $\varphi = 1$ on Ω; clearly $\varphi \in S(\Omega)$ and $V \subset \operatorname{supp}_o \varphi \subset W$. Otherwise put $f(x) = \operatorname{dist}(x, \Omega \setminus W)$ for $x \in X$. Let $R > 0$ be such that $W \subset U(0, R)$ and set $G = U(0, 2R)$. Property (LA_1) (via Remark 20) implies that there is a Lipschitz $g \in C^1(G)$ such that $|f(x) - g(x)| \leq \frac{\delta}{3}$ whenever $x \in G$. By property (iii) of the partition ring $S(G)$ used with $\varepsilon = \frac{\delta}{3}$ there is $h \in S(G)$ such that $h = 0$ on $(G \cap \Omega) \setminus W$ and $h = 1$ on V. Now put $\varphi = h$ on $\Omega \cap G$ and $\varphi = 0$ on $\Omega \setminus G$. Then it is easily seen that $\varphi \in S(\Omega)$ (the fact that φ is Lipschitz may be seen similarly as in the proof of property (ii) of the partition ring). Clearly, $V \subset \operatorname{supp}_o \varphi \subset W$.

□

2.4. **Whitney C^1 conditions.** The classical Whitney's extension condition for C^k-smooth functions from [Wh1] can be in the C^1 case easily reformulated in the "coordinate free" terms and this formulation directly generalises to the infinite-dimensional case. Namely, if X, Y are normed linear spaces and $A \subset X$ is a closed set, then $f\colon A \to Y$ satisfies Whitney's ("original") C^1 extension condition, if:

($\mathrm{W_{or}}$) There exists a continuous mapping $G\colon A \to \mathcal{L}(X; Y)$ such that for each $x \in A$ and $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|f(z) - f(y) - G(y)[z - y]\| \leq \varepsilon\|z - y\|$$
whenever $y, z \in U(x, \delta) \cap A$.

To reformulate condition ($\mathrm{W_{or}}$) to a more natural form we will use the following notion.

Definition 23. Let X, Y be normed linear spaces, $A \subset X$, $f\colon A \to Y$, and $x \in A$. We say that $L \in \mathcal{L}(X; Y)$ is a strict derivative of f at x with respect to A (resp. a strict derivative of f at x) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|f(z) - f(y) - L(z - y)\| \leq \varepsilon\|z - y\|$$
whenever $y, z \in U(x, \delta) \cap A$ (resp. $y, z \in U(x, \delta)$).

Remark 24.

(a) Let X, Y be normed linear spaces, $A \subset X$, and $f\colon A \to Y$. Clearly $L \in \mathcal{L}(X; Y)$ is a strict derivative of f at $x \in A$ with respect to A if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that the mapping $f - L$ is ε-Lipschitz on $U(x, \delta) \cap A$.

(b) If L is a strict derivative of f at x, then clearly f is Fréchet differentiable at x and $Df(x) = L$. On the other hand, for some A it can occur that f has more than one strict derivative at x with respect to A.

(c) It is well known that if $\Omega \subset X$ is open and $f \in C^1(\Omega; Y)$, then $Df(x)$ is a strict derivative of f at x for each $x \in \Omega$ (cf. [M, p. 19]).

Definition 25. Let X, Y be normed linear spaces, $A \subset X$, and $f\colon A \to Y$. We say that f satisfies condition (W) if there exists a continuous mapping $G\colon A \to \mathcal{L}(X; Y)$ such that for each $x \in A$ the linear mapping $G(x)$ is a strict derivative of f at x with respect to A. In this case we will say that f satisfies condition (W) with G. If f is defined on a set larger than A, then we say that f satisfies condition (W) on A if the restriction $f \restriction_A$ satisfies condition (W).

The following basic easy fact follows from Remark 24(c).

Fact 26. *Let X, Y be normed linear spaces, $\Omega \subset X$ open, $A \subset \Omega$, and $f\colon A \to Y$. If f can be extended to a mapping $g \in C^1(\Omega; Y)$, then f satisfies condition (W) with $G = Dg\restriction_A$.*

Remark 27. Let X, Y be normed linear spaces, $A \subset X$, $f\colon A \to Y$, and $g \in C^1(\Omega; Y)$ for some open $\Omega \supset A$. If f satisfies condition (W) with G, then $f + g\restriction_A$ satisfies condition (W) with $G + Dg\restriction_A$. This follows from Remark 24(c) and the (obvious) additivity of (relative) strict derivatives.

In [JS2] the authors use condition (W) (without speaking about strict derivatives or Whitney's condition) and call it "the mean value condition". Our notation comes from the easy fact that conditions ($\mathrm{W_{or}}$) and (W) are equivalent. Indeed it easily follows from the inequality
$$\big|\|f(z) - f(y) - G(y)[z - y]\| - \|f(z) - f(y) - G(x)[z - y]\|\big| \leq \|G(y) - G(x)\| \cdot \|z - y\|$$
and the continuity of G (which is assumed both in (W) and ($\mathrm{W_{or}}$)).

Note that by Fact 26 condition (W) is a necessary condition for the existence of a C^1-smooth extension so Theorem W (from Introduction) could be stated in the form of an equivalence.

Further note that condition (W) is not easily verifiable, since it postulates the existence of a mapping G; despite this Theorem W is an important result with many applications. If the set A is in some sense "thick" at each of its points, then f can have at each $x \in A$ at most one strict derivative with respect to A and so (W) holds if and only if f has a strict derivative $L(x)$ at each point $x \in A$ with respect to A and the mapping $x \mapsto L(x)$ is continuous. (Note that this continuity is not automatic, see [KZ, Example 4.14].)

In [JS1], which deals with the case $Y = \mathbb{R}$, the authors use a condition (called (E) there) which is equivalent to (W) (it is clearly weaker than (W) and the opposite implication follows from the Bartle-Graves selection theorem; cf. [AFK1, Lemma 2]).

2.5. **C^1 extension theorems from [JS1] and [JS2].** The first published infinite-dimensional Whitney-type C^1 extension theorem [JS1, Theorem A.2] generalises Theorem W to the case of real functions on a Banach space X which satisfies approximation condition (A) (see Remark 17), which is called (*) in [JS1, Theorem A.2]. Note that statement (b) of Theorem W is not formulated in [JS1, Theorem A.2], but it is contained implicitly in its proof. Moreover, [JS1, Theorem A.2] gives conditions under which there exists a Lipschitz C^1-smooth extension, and so answers a question which was not considered in [Wh1].

Note that the main result of [JS1] (Theorem A.2) is contained in the appendix to the first part of the article (which deals with the extensions from linear subspaces) and the proofs in the appendix are not given fully, but rather there is only an outlined list of changes to the corresponding (simpler) proofs in the first part. This makes the appendix very hard to follow and verify.

In [JS2] the authors formulate [JS2, Theorem 3.1], which extends Theorem W (without statement (b)) for mappings between Banach spaces X and Y such that the pair (X, Y) satisfies the following condition:

(*) There exists C_0 such that for every $A \subset X$, for every L-Lipschitz $f : A \to Y$, and every $\varepsilon > 0$ there is a C^1-smooth and $C_0 L$-Lipschitz $g : X \to Y$ such that $\|f(x) - g(x)\| < \varepsilon$ for all $x \in A$.

However, the proof of [JS2, Theorem 3.1] contains a serious flaw. Namely, in the proof of Lemma 3.8 on page 1213, lines 8, 7 from below, the number $\|h(x) - \Delta^n_\beta(x)\|$ is estimated from above by $\frac{\varepsilon'}{2^{n+2}L_{n,\beta(n)}}$. If $x \in A$, $(n, 0) \in F_x$ and $\beta(n) = 0$, then this estimate follows from (3.5) and the inequality on page 1212, line 8 from below (which holds for $z \in A$) used with $z := x$. But we see no possibility to obtain the above estimate in the case when $x \notin A$, $(n, 0) \in F_x$, and $\beta(n) = 0$, or to show that this case is impossible. It seems that this flaw is related to the fact that Lemma 3.6(iii) is not used anywhere in the proofs.

Our Theorem 34 shows that the generalisation of Theorem W holds if we suppose that the pair (X, Y) satisfies conditions (LA_1) and (LE) (called (E) in [JS2]). The conjunction (LE) & (LA_1) is very close to condition (*) from [JS2]. Indeed, the fact that (LE) & (LA_1) implies (*) is trivial. On the other hand, the converse implication holds if Y is a dual space (see [JS2, Remark 1.3(2)]; recall that (LA_1) is equivalent to property (A) from [JS2]). Nevertheless, it is not known whether the equivalence holds in general. We do not know whether [JS2, Theorems 3.1, 3.2] hold, however we do not see any way how to prove these theorems without using some assumption on extendability of Lipschitz mappings. On the other hand recall that we believe that the proofs in [JS2] can be modified to correctly prove Theorem 1 (which uses the assumption (LE) & (LA_1) instead of (*)). We also note that whenever the authors of [JS2] prove that a concrete pair of spaces satisfies (*), they do it via properties (LE) and (LA_1).

2.6. **Pairs (X, Y) for which C^1 extension theorems hold.** Let X, Y be normed linear spaces. Consider the following statements ("basic C^1 Whitney extension theorem", resp. "Lipschitz C^1 Whitney extension theorem"):

BW: For each closed set $A \subset X$ and each mapping $f : A \to Y$ which satisfies condition (W) there exists $g \in C^1(X; Y)$ which extends f.

LW: There is $C > 0$ such that for each closed set $A \subset X$ and each L-Lipschitz mapping $f : A \to Y$ which satisfies condition (W) with G satisfying $\sup_{x \in A} \|G(x)\| \leq L$ there exists a CL-Lipschitz $g \in C^1(X; Y)$ which extends f.

We present here some facts concerning the following properties of (X, Y) (where X, Y are non-trivial normed linear spaces):

(a) Statement BW holds for (X, Y).
(b) Statement LW holds for (X, Y).
(c) The pair (X, Y) has both properties (LE) and (LA_1).
(d) Condition (*) (see Subsection 2.5) holds for (X, Y).

We start with the following almost obvious remark.

Remark 28.

(i) If we equip X and Y with equivalent norms, the validity of any of (a)–(d), resp. (LE), resp. (LA_1), does not change.
(ii) If one of statements (a)–(d), resp. (LE), resp. (LA_1), holds both for (X, Y_1) and (X, Y_2), then it holds for $(X, Y_1 \oplus_\infty Y_2)$ as well.

Recall that Theorem 1 gives (c)$\Rightarrow$(a) and (c)$\Rightarrow$(b). Recall also that (c)$\Rightarrow$(d), and (c)$\Leftrightarrow$(d) if Y is a dual space (see the last paragraph of Subsection 2.5). Further, [JS2, Proposition 2.8] immediately gives that (b)$\Rightarrow$(d). So, if Y is a dual space, then (b)$\Leftrightarrow$(c). Thus, the pairs (X, Y) for which statement LW holds are "almost characterised". But the (more interesting) case of statement BW is more difficult. By a standard method (cf. [JS1, Corollary A.4 and the note after it] or [S, Proposition 4.3.10]) we obtain the following interesting fact: If (a) holds, then X is an Asplund space. Indeed, suppose that (a) holds and choose $y \in Y$ and $\phi \in Y^*$ such that $\phi(y) \neq 0$. Set $A = \{0\} \cup (X \setminus U_X)$ and define $f : A \to Y$ by $f(0) = y$ and $f(x) = 0$ for $x \in A \setminus \{0\}$. Since f clearly satisfies condition (W), by (a) there is $g \in C^1(X; Y)$ that extends f. Then $\phi \circ g$ is a C^1-smooth bump on X and so X is an Asplund space ([DGZ, Theorem II.5.3]).

Consequently, if X is separable, then the following statements are equivalent:

(i) Statement BW holds for the pair $(X, \mathbb{R})$.
(ii) X^* is separable.
(iii) The space X has property (LA_1).

Indeed, we have just proved (i)⇒(ii) and (ii)⇒(iii) is mentioned at the end of Example 21. Finally, (iii)⇒(i) follows from Theorem 1, since $(X, \mathbb{R})$ always has property (LE) (or from [JS1, Theorem A.2] and Remark 17).

However, for non-separable X no interesting condition equivalent to (i) is known.

The examples of pairs (X, Y) that satisfy both (LE) and (LA_1) are given in Example 43. These cover (and generalise) all the cases presented in [JS2, Corollary 3.4] and [JS1, p. 174], as well as some others not mentioned in [JS2] or [JS1] (items (d)–(f)).

Finally, we remark that we do not know anything concerning the validity of any of the statements (a)–(d) e.g. for the pairs (c_0, ℓ_2), $\bigl(L_2([0,1]), L_1([0,1])\bigr)$, or (H, H), where H is a non-separable Hilbert space. For any pair (ℓ_p, ℓ_q), $1 < p < q < \infty$, we know that (c) (and consequently also (b) and (d)) does not hold (Example 9) but we do not know whether (a) holds.

3. Proof of the basic extension results

In this section we prove the basic C^1 extension theorems (Theorem 34 and 35) for mappings defined on arbitrary closed sets as a consequence of two propositions with more complicated assumptions, which will be used later for extension results from special sets (which do not follow from these "basic" versions).

The following "mixing lemma" is central to the proof of the main Lemma 30. Its idea is implicitly contained in [JS1, Proof of Lemma 2.3].

Lemma 29. *Let X, Y be normed linear spaces such that X has property (LA_1) with $C \geq 1$. Let $V \subset X$ be open, $K \geq 0$, and let $u, v \in C^1(V; Y)$ be K-Lipschitz. Let $E \subset V$. Then for each positive $\varepsilon > \sup_{x\in E} \|v(x) - u(x)\|$ there is $g \in C^1(V; Y)$ such that $g = v$ and $Dg = Dv$ on E, $\|g(x) - u(x)\| \leq \varepsilon$ for each $x \in V$, and $\|Dg(x)\| \leq 4CK$ for each $x \in V$.*

Proof. Without loss of generality assume that $E \neq \emptyset$. Let $\zeta = \sup_{x\in E} \|v(x) - u(x)\|$ and set $\delta = \frac{\varepsilon-\zeta}{2} > 0$. By [HJ, Theorem 7.86] there is $\mu \in C^1(V)$ that is $3CK$-Lipschitz and such that $\sup_{x\in V} \bigl|\mu(x) - \bigl(\|v(x) - u(x)\| + \frac{\delta}{4}\bigr)\bigr| \leq \frac{\delta}{4}$, i.e. $\|v(x) - u(x)\| \leq \mu(x) \leq \|v(x) - u(x)\| + \frac{\delta}{2}$ for each $x \in V$. Further, set $\varphi(t) = \frac{1}{\delta}\int_t^{t+\delta} \omega(s)\,\mathrm{d}s$ for $t \in \mathbb{R}$, where $\omega(t) = 1$ for $t \leq \varepsilon$ and $\omega(t) = \frac{\varepsilon}{t}$ for $t \geq \varepsilon$. Then clearly $\varphi \in C^1(\mathbb{R})$, $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $t \leq \zeta + \delta$, and $\varphi(t) \leq \frac{\varepsilon}{t}$ for $t > 0$. Moreover,

$$|\varphi'(t)| \leq \tfrac{1}{t} \text{ for } t \geq \varepsilon \text{ and } |\varphi'(t)| \leq \tfrac{1}{\varepsilon} \text{ for } t \leq \varepsilon.$$

Indeed, $\varphi'(t) = \frac{1}{\delta}\bigl(\omega(t+\delta) - \omega(t)\bigr)$. By distinguishing the cases $t \leq \varepsilon - \delta$, $\varepsilon - \delta < t < \varepsilon$, and $t \geq \varepsilon$ we can explicitly compute $|\varphi'(t)|$ and easily obtain the required inequalities. Finally, set $\psi = \varphi \circ \mu$ and $g = u + \psi \cdot (v - u)$. Obviously, $g \in C^1(V; Y)$. Since $\mu(x) \leq \zeta + \frac{\delta}{2}$ for $x \in E$, it follows that $g = v$ on a neighbourhood of E and hence $Dg = Dv$ on E. Next, $\|g(x) - u(x)\| \leq \psi(x)\|v(x) - u(x)\| \leq \frac{\varepsilon}{\mu(x)}\|v(x) - u(x)\| \leq \varepsilon$ whenever $x \in V$ is such that $u(x) \neq v(x)$ (and clearly $\|g(x) - u(x)\| = 0$ whenever $u(x) = v(x)$).

Finally, let $x \in V$. Then $Dg(x) = Du(x) + \varphi'(\mu(x))D\mu(x) \cdot \bigl(v(x) - u(x)\bigr) + \psi(x)\bigl(Dv(x) - Du(x)\bigr)$. If $\mu(x) \geq \varepsilon$, then $|\varphi'(\mu(x))| \cdot \|v(x) - u(x)\| \leq \frac{1}{\mu(x)}\mu(x) = 1$, otherwise $|\varphi'(\mu(x))| \cdot \|v(x) - u(x)\| \leq \frac{1}{\varepsilon}\mu(x) < 1$. Therefore

$$\|Dg(x)\| \leq \bigl(1 - \psi(x)\bigr)\|Du(x)\| + 1 \cdot \|D\mu(x)\| + \psi(x)\|Dv(x)\| \leq K + 3CK \leq 4CK.$$

□

The proof of the following main lemma is based on a combination of ideas of proofs of [HJ, Proposition 7.94] (where the role of G is played by Df) and [HJ, Theorem 7.86] (these results come originally from [HJ1]) together with the mixing lemma Lemma 29.

Lemma 30. *Let X, Y be normed linear spaces such that the pair (X, Y) has property (LA_1) with $C \geq 1$. Let $\Omega \subset X$ be open, let $A \subset \Omega$ be relatively closed and suppose that the pair (A, Y) has property (LLE). Let $L \geq 0$ and let $f : \Omega \to Y$ be an L-Lipschitz mapping that satisfies condition (W) on A with G such that $\sup_{x\in A}\|G(x)\| \leq L$. Then for any $\varepsilon > 0$ there is an $8C^2L$-Lipschitz mapping $g \in C^1(\Omega; Y)$ such that $\|f(x) - g(x)\| \leq \varepsilon$ for all $x \in \Omega$, $(f - g)\restriction_A$ is ε-Lipschitz, and $\|G(x) - Dg(x)\| \leq \varepsilon$ for all $x \in A$.*

Proof. Let $\varepsilon > 0$ and without loss of generality assume that $L > 0$ and $\varepsilon \leq L$. For each $x \in \Omega \setminus A$ find $r(x) > 0$ such that $U(x, 2r(x)) \subset \Omega \setminus A$. For each $x \in A$ find $K \geq 1$ from property (LLE) and find $\delta > 0$ such that $U(x, \delta) \subset \Omega$, $f - G(x)$ is $\frac{\varepsilon}{6CK}$-Lipschitz on $U(x, \delta) \cap A$ (Remark 24(a)), and $\|G(y) - G(x)\| < \frac{\varepsilon}{3}$ for each $y \in U(x, \delta) \cap A$. By property (LLE) there is a neighbourhood U of x such that $U \subset U(x, \delta)$ and the restriction of the mapping $f - G(x)$ to $U \cap A$ has an $\frac{\varepsilon}{6C}$-Lipschitz extension to U. Let $r(x) > 0$ be such that $U(x, 2r(x)) \subset U$. Then the restriction of the mapping $f - G(x)$ to $U(x, 2r(x)) \cap A$ has an $\frac{\varepsilon}{6C}$-Lipschitz extension to $U(x, 2r(x))$. Note that

$$\|G(y) - G(x)\| < \frac{\varepsilon}{3} \quad \text{for each } y \in U(x, 2r(x)) \cap A. \tag{1}$$

By Lemma 22 there is a locally finite and σ-discrete C^1-smooth Lipschitz partition of unity on Ω subordinated to $\{U(x, r(x));\ x \in \Omega\}$. We may assume that the partition of unity is of the form $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, where for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}_o \psi_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω. Set $\Gamma = \mathbb{N} \times \Lambda$. Given $\gamma \in \Gamma$ note that

$$D\psi_\gamma(x) = 0 \text{ whenever } x \in \Omega \setminus \operatorname{supp}_o \psi_\gamma, \tag{2}$$

since if $\psi_\gamma(x) = 0$, then ψ_γ (which is non-negative) attains its minimum at x. For each $\gamma \in \Gamma$ let $U_\gamma = U(x_\gamma, r(x_\gamma))$ be such that $\operatorname{supp}_o \psi_\gamma \subset U_\gamma$ and denote $V_\gamma = U(x_\gamma, 2r(x_\gamma))$. Let $L_\gamma \geq 1$ be a Lipschitz constant of ψ_γ. For $x \in \Omega$ denote $S_x = \{\gamma \in \Gamma;\ x \in \operatorname{supp}_o \psi_\gamma\}$ and note that if $\gamma \in S_x$, then $x \in U_\gamma$. Since for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}_o \psi_{n\alpha}\}_{\alpha\in\Lambda}$ is disjoint, there exists $M_x \subset \mathbb{N}$ and $\alpha_x\colon M_x \to \Lambda$ such that

$$S_x = \bigl\{(n, \alpha_x(n));\ n \in M_x\bigr\}. \tag{3}$$

Fix $\gamma = (n, \alpha) \in \Gamma$. Since the pair (X, Y) has property (LA$_1$), there is a CL-Lipschitz mapping $u_\gamma \in C^1(V_\gamma; Y)$ such that

$$\|f(x) - u_\gamma(x)\| \leq \frac{\varepsilon}{12 \cdot 2^n L_\gamma} \quad \text{for each } x \in V_\gamma. \tag{4}$$

If $x_\gamma \notin A$, then we set $g_\gamma = u_\gamma$. Now we deal with the case $x_\gamma \in A$. By the definition of $r(x_\gamma)$ there is an $\frac{\varepsilon}{6C}$-Lipschitz mapping $f_\gamma\colon V_\gamma \to Y$ such that $f_\gamma = f - G(x_\gamma)$ on $V_\gamma \cap A$. By property (LA$_1$) there is an $\frac{\varepsilon}{6}$-Lipschitz mapping $\bar{v}_\gamma \in C^1(V_\gamma; Y)$ such that $\|f_\gamma(x) - \bar{v}_\gamma(x)\| \leq \frac{\varepsilon}{12 \cdot 2^n L_\gamma}$ for each $x \in V_\gamma$. Put $v_\gamma = \bar{v}_\gamma + G(x_\gamma)$ and note that $f - v_\gamma = f_\gamma - \bar{v}_\gamma$ on $V_\gamma \cap A$. Then, using also (4), we obtain $\|v_\gamma(x) - u_\gamma(x)\| = \|v_\gamma(x) - f(x) + f(x) - u_\gamma(x)\| \leq \|\bar{v}_\gamma(x) - f_\gamma(x)\| + \|f(x) - u_\gamma(x)\| \leq \frac{\varepsilon}{6 \cdot 2^n L_\gamma}$ for every $x \in V_\gamma \cap A$. Therefore by Lemma 29 (used on CL-Lipschitz u_γ and $(L + \frac{\varepsilon}{6})$-Lipschitz v_γ, the set $E = V_\gamma \cap A$, and $K = \frac{5}{4}CL$) there is $g_\gamma \in C^1(V_\gamma; Y)$ which is $5C^2L$-Lipschitz (note that V_γ is convex) and such that $g_\gamma = v_\gamma$ and $Dg_\gamma = Dv_\gamma$ on $V_\gamma \cap A$ and

$$\|g_\gamma(x) - u_\gamma(x)\| \leq \frac{\varepsilon}{4 \cdot 2^n L_\gamma} \quad \text{for each } x \in V_\gamma. \tag{5}$$

Since $f - g_\gamma = f - v_\gamma = f_\gamma - \bar{v}_\gamma$ on $V_\gamma \cap A$, where f_γ is $\frac{\varepsilon}{6C}$-Lipschitz and $\bar{v}_\gamma$ is $\frac{\varepsilon}{6}$-Lipschitz, it follows that

$$f - g_\gamma \text{ is } \frac{\varepsilon}{3}\text{-Lipschitz on } V_\gamma \cap A. \tag{6}$$

Further, $Dg_\gamma - G(x_\gamma) = Dv_\gamma - G(x_\gamma) = D\bar{v}_\gamma$ on $V_\gamma \cap A$ and hence

$$\|Dg_\gamma(x) - G(x_\gamma)\| \leq \frac{\varepsilon}{6} \quad \text{for each } x \in V_\gamma \cap A. \tag{7}$$

Finally, in both cases (i.e. $x_\gamma \in A$, resp. $x_\gamma \notin A$) using (4) and (5) we obtain that

$$\|f(x) - g_\gamma(x)\| \leq \|f(x) - u_\gamma(x)\| + \|u_\gamma(x) - g_\gamma(x)\| \leq \frac{\varepsilon}{3 \cdot 2^n L_\gamma} \quad \text{for each } x \in V_\gamma. \tag{8}$$

Note that both (6) and (7) hold trivially also in the case $x_\gamma \notin A$, since then $V_\gamma \cap A = \emptyset$.

Define $\bar{g}_\gamma\colon \Omega \to Y$ by $\bar{g}_\gamma = g_\gamma$ on V_γ and $\bar{g}_\gamma = 0$ on $\Omega \setminus V_\gamma$. Finally, we define the mapping $g\colon \Omega \to Y$ by

$$g = \sum_{\gamma\in\Gamma} \psi_\gamma \bar{g}_\gamma.$$

Since $\{\operatorname{supp}_o \psi_\gamma\}_{\gamma\in\Gamma}$ is locally finite, given $x \in \Omega$ there exists $\delta > 0$ and a finite $F \subset \Gamma$ such that $\psi_\gamma(y) = 0$ for each $\gamma \in \Gamma \setminus F$ and $y \in U(x, \delta)$. Set $F_x = \{\gamma \in F;\ x \in V_\gamma\}$. Then there exists $0 < \delta_x \leq \delta$ such that $\psi_\gamma(y) = 0$ for each $\gamma \in \Gamma \setminus F_x$ and $y \in U(x, \delta_x)$, and $U(x, \delta_x) \subset V_\gamma$ for each $\gamma \in F_x$. Indeed, $\operatorname{dist}(x, U_\gamma) \geq r(x_\gamma)$ whenever $\gamma \in F \setminus F_x$ and each V_γ is open. It follows that g is well-defined and $g = \sum_{\gamma\in F_x} \psi_\gamma \bar{g}_\gamma = \sum_{\gamma\in F_x} \psi_\gamma g_\gamma$ on $U(x, \delta_x)$. Consequently, $g \in C^1(\Omega; Y)$ and

$$Dg(x) = \sum_{\gamma\in F_x} D(\psi_\gamma g_\gamma)(x) = \sum_{\gamma\in F_x} \psi_\gamma(x) Dg_\gamma(x) + D\psi_\gamma(x) \cdot g_\gamma(x) = \sum_{\gamma\in S_x} \psi_\gamma(x) Dg_\gamma(x) + D\psi_\gamma(x) \cdot g_\gamma(x) \tag{9}$$

by (2) (note that $S_x \subset F_x$). Further, since $1 = \sum_{\gamma\in\Gamma} \psi_\gamma = \sum_{\gamma\in F_x} \psi_\gamma$ on $U(x, \delta_x)$, it follows that $\sum_{\gamma\in F_x} D\psi_\gamma = D\sum_{\gamma\in F_x} \psi_\gamma = 0$ on $U(x, \delta_x)$. Hence, using also (2), we obtain

$$\sum_{\gamma\in S_x} D\psi_\gamma(x) = 0 \quad \text{for each } x \in \Omega. \tag{10}$$

Now choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\|f(x) - g(x)\| = \left\|\sum_{\gamma\in\Gamma} \psi_\gamma(x)\bigl(f(x) - \bar{g}_\gamma(x)\bigr)\right\| \leq \sum_{\gamma\in S_x} \psi_\gamma(x)\|f(x) - g_\gamma(x)\| \leq \varepsilon \sum_{\gamma\in S_x} \psi_\gamma(x) = \varepsilon,$$

where the last inequality follows from (8).

Next we show that $(f - g)\restriction_A$ is ε-Lipschitz and g is $8C^2L$-Lipschitz. Let $x, y \in \Omega$. Denote $h_\gamma = f - \bar{g}_\gamma$ for short. Then

$$(f - g)(x) - (f - g)(y) = \sum_{\gamma\in\Gamma} \psi_\gamma(x)(f - \bar{g}_\gamma)(x) - \sum_{\gamma\in\Gamma} \psi_\gamma(y)(f - \bar{g}_\gamma)(y) = \sum_{\gamma\in S_x\cup S_y} \bigl(\psi_\gamma(x)h_\gamma(x) - \psi_\gamma(y)h_\gamma(y)\bigr).$$

Let us estimate the norm of the last sum. For any $\gamma \in \Gamma$ the following holds:

$$\begin{aligned} \big\|\psi_\gamma(x)h_\gamma(x)-\psi_\gamma(y)h_\gamma(y)\big\| &\le \big\|\psi_\gamma(x)h_\gamma(x)-\psi_\gamma(x)h_\gamma(y)\big\| + \big\|\psi_\gamma(x)h_\gamma(y)-\psi_\gamma(y)h_\gamma(y)\big\| \\ &= \psi_\gamma(x)\|h_\gamma(x)-h_\gamma(y)\| + |\psi_\gamma(x)-\psi_\gamma(y)|\|h_\gamma(y)\| \\ &\le \psi_\gamma(x)\|h_\gamma(x)-h_\gamma(y)\| + L_\gamma\|x-y\|\|h_\gamma(y)\|. \end{aligned} \tag{11}$$

Let $\gamma = (n,\alpha) \in S_x \cup S_y$. If $\gamma \in S_y \setminus S_x$, then $\big\|\psi_\gamma(x)h_\gamma(x)-\psi_\gamma(y)h_\gamma(y)\big\| \le \frac{\varepsilon}{3\cdot 2^n}\|x-y\|$ by (11) and (8), as $\psi_\gamma(x)=0$ and $\bar{g}_\gamma(y) = g_\gamma(y)$. It is easily seen that by the symmetry the same estimate holds if $\gamma \in S_x \setminus S_y$. If $\gamma \in S_x \cap S_y$, then $x, y \in U_\gamma$ and we use the fact that g_γ is $5C^2L$-Lipschitz on U_γ and so h_γ is $(5C^2+1)L$-Lipschitz on U_γ. Consequently, by (11) and (8),

$$\big\|\psi_\gamma(x)h_\gamma(x)-\psi_\gamma(y)h_\gamma(y)\big\| \le \Big(\psi_\gamma(x)(5C^2+1)L + \frac{\varepsilon}{3\cdot 2^n}\Big)\|x-y\|. \tag{12}$$

If moreover $x, y \in A$, then $x, y \in U_\gamma \cap A$, and so $\|h_\gamma(x)-h_\gamma(y)\| \le \frac{\varepsilon}{3}\|x-y\|$ by (6). Hence $\big\|\psi_\gamma(x)h_\gamma(x)-\psi_\gamma(y)h_\gamma(y)\big\| \le \big(\psi_\gamma(x)\frac{\varepsilon}{3} + \frac{\varepsilon}{3\cdot 2^n}\big)\|x-y\|$.

Putting this all together, for $x, y \in A$ the above estimates together with (3) yield

$$\begin{aligned} \big\|(f-g)(x)-(f-g)(y)\big\| &\le \Bigg(\sum_{(n,\alpha)\in S_x\setminus S_y}\frac{\varepsilon}{3\cdot 2^n} + \sum_{(n,\alpha)\in S_y\setminus S_x}\frac{\varepsilon}{3\cdot 2^n} + \sum_{(n,\alpha)\in S_x\cap S_y}\Big(\psi_{n\alpha}(x)\frac{\varepsilon}{3}+\frac{\varepsilon}{3\cdot 2^n}\Big)\Bigg)\|x-y\| \\ &= \Bigg(\sum_{(n,\alpha)\in S_x}\frac{\varepsilon}{3\cdot 2^n} + \sum_{(n,\alpha)\in S_y\setminus S_x}\frac{\varepsilon}{3\cdot 2^n} + \sum_{(n,\alpha)\in S_x\cap S_y}\psi_{n\alpha}(x)\frac{\varepsilon}{3}\Bigg)\|x-y\| \\ &\le \Bigg(\sum_{n\in M_x}\frac{\varepsilon}{3\cdot 2^n} + \sum_{n\in M_y}\frac{\varepsilon}{3\cdot 2^n} + \frac{\varepsilon}{3}\sum_{\gamma\in\Gamma}\psi_\gamma(x)\Bigg)\|x-y\| \le \varepsilon\|x-y\|. \end{aligned}$$

In the case of general $x, y \in \Omega$, this time using (12) we analogously obtain that $f-g$ is $(5C^2+2)L$-Lipschitz (recall that $\varepsilon \le L$) and consequently g is $(5C^2+3)L$-Lipschitz and hence also $8C^2L$-Lipschitz.

Finally, to estimate the distance between G and Dg fix $x \in A$. Then

$$\begin{aligned} \|G(x)-Dg(x)\| &= \Bigg\|\sum_{\gamma\in S_x}\psi_\gamma(x)G(x)-\psi_\gamma(x)Dg_\gamma(x)-D\psi_\gamma(x)\cdot g_\gamma(x)\Bigg\| \\ &= \Bigg\|\sum_{\gamma\in S_x}\psi_\gamma(x)\big(G(x)-Dg_\gamma(x)\big) + \sum_{\gamma\in S_x}D\psi_\gamma(x)\cdot\big(f(x)-g_\gamma(x)\big)\Bigg\| \\ &\le \sum_{\gamma\in S_x}\psi_\gamma(x)\big\|G(x)-Dg_\gamma(x)\big\| + \sum_{n\in M_x}\|D\psi_{n\alpha_x(n)}(x)\|\,\big\|f(x)-g_{n\alpha_x(n)}(x)\big\| \\ &\le \sum_{\gamma\in S_x}\psi_\gamma(x)\big(\|G(x)-G(x_\gamma)\| + \|G(x_\gamma)-Dg_\gamma(x)\|\big) + \sum_{n\in M_x}L_{n\alpha_x(n)}\frac{\varepsilon}{3\cdot 2^n L_{n\alpha_x(n)}} \\ &< \Big(\frac{\varepsilon}{3}+\frac{\varepsilon}{6}\Big)\sum_{\gamma\in S_x}\psi_\gamma(x) + \frac{\varepsilon}{3} < \varepsilon, \end{aligned}$$

where the first equality follows from (9), the second one from (10), the first inequality follows from (3), the second one from (8), and the third one from (1) and (7).

□

Remark 31. It is not too difficult to see that a slight modification of the proof of the previous lemma shows that it holds also if the property (LLE) of the pair (A, Y) is weakened to the following property (△):

> For every $x \in A$ there is $K \ge 1$ such that for each $\delta > 0$ and each Q-Lipschitz mapping $h\colon U(x,\delta)\cap A \to Y$ there is $0 < \eta \le \delta$ such that for each $\varepsilon > 0$ there is a KQ-Lipschitz mapping $\bar{h}\colon U(x,\eta) \to Y$ satisfying $\|\bar{h}(y)-h(y)\| \le \varepsilon$ whenever $y \in U(x,\eta)\cap A$.

Consequently, the assumption of (LLE) in the next proposition can also be weakened to the property above. We have however no application of this weaker assumption so we decided to not use it.

Note also that if a pair of spaces (X, Y) has property (*) from [JS2], then for every $A \subset X$ the pair (A, Y) has property (△) above (and the pair (X, Y) also has (LA_1)). So in both Lemma 30 above and Proposition 32 below the conjunction of assumptions (LA_1) and (LLE) can be replaced by property (*) from [JS2]. Note however that Proposition 32 requires another assumption "(a)" (of global extendability from A) which probably does not follow from property (*) (in case of non-dual spaces Y).

Proposition 32. *Let X be a normed linear space and Y a Banach space such that the pair (X, Y) has property (LA_1) with $C \ge 1$. Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed such that each component of Ω has a non-empty intersection with A. Suppose that*

(a) every Q-Lipschitz mapping from A to Y can be extended to a CQ-Lipschitz mapping on Ω and
(b) the pair (A, Y) has property (LLE).

Let $L \geq 0$ and let $f\colon A \to Y$ be an L-Lipschitz mapping that satisfies condition (W) with G such that $\sup_{x\in A}\|G(x)\| \leq L$. Then f can be extended to a $16C^3L$-Lipschitz mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A.

Proof. Without loss of generality we assume that $L > 0$. By recursion we will construct a sequence of mappings $g_k \in C^1(\Omega; Y)$, $k \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ the following hold:

(i) g_n is $\frac{16C^3L}{2^n}$-Lipschitz,
(ii) $\left\| f(x) - \sum_{k=1}^n g_k(x)\right\| \leq \frac{L}{2^n}$ for each $x \in A$,
(iii) the mapping $f - \sum_{k=1}^n g_k\restriction_A$ is $\frac{L}{2^n}$-Lipschitz,
(iv) $\left\|G(x) - \sum_{k=1}^n Dg_k(x)\right\| \leq \frac{L}{2^n}$ for each $x \in A$.

For the first step we apply Lemma 30 to a CL-Lipschitz extension of f to Ω (using the assumption (a)) with $\varepsilon = \frac{L}{2}$ to obtain $g_1 \in C^1(\Omega; Y)$ such that (i)–(iv) hold for $n = 1$. For the inductive step let $n > 1$ and assume that $g_1, \dots, g_{n-1}$ are defined and (i)–(iv) hold with $n-1$ in place of n. Using Remark 27 we see that the mapping $f - \sum_{k=1}^{n-1} g_k\restriction_A$ satisfies condition (W) with $\widetilde{G} = G - \sum_{k=1}^{n-1}(Dg_k)\restriction_A$. By the assumption (a) there is $F\colon \Omega \to Y$ which is a $\frac{CL}{2^{n-1}}$-Lipschitz extension of $f - \sum_{k=1}^{n-1} g_k\restriction_A$ (we use (iii) of the inductive assumption). Since $\|\widetilde{G}(x)\| \leq \frac{L}{2^{n-1}}$ for each $x \in A$, by Lemma 30 applied to F with "$L := \frac{CL}{2^{n-1}}$, $G := \widetilde{G}$", and $\varepsilon = \frac{L}{2^n}$ we obtain an $\frac{8C^3L}{2^{n-1}}$-Lipschitz mapping $g_n \in C^1(\Omega; Y)$ such that $\left\| f(x) - \sum_{k=1}^n g_k(x)\right\| = \|F(x) - g_n(x)\| \leq \frac{L}{2^n}$ for all $x \in A$, $f - \sum_{k=1}^n g_k\restriction_A = (F - g_n)\restriction_A$ is $\frac{L}{2^n}$-Lipschitz, and $\left\|G(x) - \sum_{k=1}^n Dg_k(x)\right\| = \|\widetilde{G}(x) - Dg_n(x)\| \leq \frac{L}{2^n}$ for all $x \in A$, and so (i)–(iv) hold.

Property (i) implies that $\|Dg_n(x)\| \leq \frac{16C^3L}{2^n}$ for any $x \in \Omega$ and so the series $\sum_{n=1}^\infty Dg_n$ converges uniformly on Ω. Property (ii) implies that the series $\sum_{n=1}^\infty g_n$ converges on A. Using [D, (8.6.5)] on each component of Ω, in which we choose any $x_0 \in A$, we obtain that $\sum_{n=1}^\infty g_n$ converges on Ω and when we set $g = \sum_{n=1}^\infty g_n$, then $Dg = \sum_{n=1}^\infty Dg_n$ and so $g \in C^1(\Omega; Y)$. Further, (i) implies that g is $16C^3L$-Lipschitz, (ii) implies that $g\restriction_A = f$, and (iv) implies that $(Dg)\restriction_A = G$.

□

Proposition 33. *Let X be a normed linear space and Y a Banach space such that the pair (X, Y) has property (LA$_1$). Let $\Omega \subset X$ be open and let $A \subset \Omega$ be relatively closed such that the pair (A, Y) has property (LLE). Suppose that $f\colon A \to Y$ satisfies condition (W) with G. Then f can be extended to a mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A.*

Proof. For each $x \in A$ using the continuity of G and Remark 24(a) we find $\Delta > 0$ such that $U(x, \Delta) \subset \Omega$, G is bounded and f is Lipschitz on $A \cap U(x, \Delta)$. By property (LLE) there is $K_x \geq 1$ and an open neighbourhood $\widetilde{V}_x$ of x such that $\widetilde{V}_x \subset U(x, \Delta)$ and each Q-Lipschitz mapping from $\widetilde{V}_x \cap A$ to Y can be extended to a K_xQ-Lipschitz mapping on $\widetilde{V}_x$. Let V_x be the union of all components of $\widetilde{V}_x$ that have a non-empty intersection with A. Then $V_x \cap A = \widetilde{V}_x \cap A$ and so

$$\text{each } Q\text{-Lipschitz mapping from } V_x \cap A \text{ to } Y \text{ can be extended to a } K_xQ\text{-Lipschitz mapping on } V_x. \tag{13}$$

Let $\delta_x > 0$ be such that $U(x, 2\delta_x) \subset V_x$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda'}$ be a locally finite C^1-smooth partition of unity on Ω subordinated to the open covering $\{U(x, \delta_x);\ x \in A\} \cup \{\Omega \setminus A\}$ of Ω (Lemma 22). Set $\Lambda = \{\alpha \in \Lambda';\ \operatorname{supp}_o \varphi_\alpha \cap A \neq \emptyset\}$. For each $\alpha \in \Lambda$ choose $x_\alpha \in A$ such that $\operatorname{supp}_o \varphi_\alpha \subset U(x_\alpha, \delta_{x_\alpha})$ and denote $U_\alpha = U(x_\alpha, \delta_{x_\alpha})$ and $V_\alpha = V_{x_\alpha}$.

Now fix an arbitrary $\alpha \in \Lambda$. The pair $(A \cap V_\alpha, Y)$ has property (LLE) by Remark 12 and $f\restriction_{A\cap V_\alpha}$ satisfies condition (W) with $G\restriction_{A\cap V_\alpha}$. So, using also (13), we can apply Proposition 32 with "$\Omega := V_\alpha$, $A := A \cap V_\alpha$, $f := f\restriction_{A\cap V_\alpha}$", and $C = \max\{K_{x_\alpha}, C'\}$, where C' is the constant from property (LA$_1$). Hence there is $g_\alpha \in C^1(V_\alpha; Y)$ which is an extension of $f\restriction_{A\cap V_\alpha}$ and such that $Dg_\alpha = G$ on $A \cap V_\alpha$. Define $\bar{g}_\alpha\colon \Omega \to Y$ by $\bar{g}_\alpha = g_\alpha$ on V_α and $\bar{g}_\alpha = 0$ on $\Omega \setminus V_\alpha$.

Now put $g = \sum_{\alpha\in\Lambda} \varphi_\alpha \bar{g}_\alpha$. Since the partition of unity is locally finite, given $x \in \Omega$ there is an open neighbourhood W_x of x and a finite $F_x \subset \Lambda'$ such that $\varphi_\alpha = 0$ on W_x for $\alpha \in \Lambda' \setminus F_x$. Therefore g is well-defined and $g = \sum_{\alpha\in F_x\cap\Lambda} \varphi_\alpha \bar{g}_\alpha$ on W_x. Moreover, if $\alpha \in \Lambda$, then $\operatorname{supp}_o \varphi_\alpha \subset U_\alpha$ and $g_\alpha \in C^1(V_\alpha; Y)$, and so $\varphi_\alpha \bar{g}_\alpha \in C^1(\Omega; Y)$. It follows that $g \in C^1(\Omega; Y)$. Further, $1 = \sum_{\alpha\in\Lambda'} \varphi_\alpha = \sum_{\alpha\in F_x} \varphi_\alpha$ on W_x. It follows that

$$\sum_{\alpha\in F_x} D\varphi_\alpha(x) = 0 \quad \text{for each } x \in \Omega. \tag{14}$$

To show that g is an extension of f suppose that $x \in A$ is given. Then $\varphi_\alpha(x) = 0$ for each $\alpha \in \Lambda' \setminus \Lambda$ and for each $\alpha \in \Lambda$ such that $x \notin U_\alpha$. Hence

$$g(x) = \sum_{\substack{\alpha\in\Lambda\\ x\in U_\alpha}} \varphi_\alpha(x) g_\alpha(x) = \sum_{\substack{\alpha\in\Lambda\\ x\in U_\alpha}} \varphi_\alpha(x) f(x) = f(x) \sum_{\alpha\in\Lambda'} \varphi_\alpha(x) = f(x).$$

Also, $D\varphi_\alpha(x) = 0$ for each $\alpha \in \Lambda' \setminus \Lambda$ and for each $\alpha \in \Lambda$ such that $x \notin U_\alpha$. (Notice that $D\varphi_\alpha(x) = 0$ whenever $\varphi_\alpha(x) = 0$, since then φ_α attains its minimum in x.) Therefore, using (14), we obtain that

$$\begin{aligned} Dg(x) &= D\Bigg(\sum_{\alpha\in F_x\cap\Lambda} \varphi_\alpha \bar{g}_\alpha\Bigg)(x) = \sum_{\alpha\in F_x\cap\Lambda} D(\varphi_\alpha \bar{g}_\alpha)(x) = \sum_{\alpha\in F_x\cap\Lambda} \big(D\varphi_\alpha(x)\cdot \bar{g}_\alpha(x) + \varphi_\alpha(x) D\bar{g}_\alpha(x)\big) \\ &= \sum_{\substack{\alpha\in F_x\cap\Lambda\\ x\in U_\alpha}} \big(D\varphi_\alpha(x)\cdot \bar{g}_\alpha(x) + \varphi_\alpha(x) D\bar{g}_\alpha(x)\big) = \sum_{\substack{\alpha\in F_x\cap\Lambda\\ x\in U_\alpha}} \big(D\varphi_\alpha(x)\cdot g_\alpha(x) + \varphi_\alpha(x) Dg_\alpha(x)\big) \\ &= \sum_{\alpha\in F_x} \big(D\varphi_\alpha(x)\cdot f(x) + \varphi_\alpha(x) G(x)\big) = \Bigg(\sum_{\alpha\in F_x} D\varphi_\alpha(x)\Bigg) f(x) + \Bigg(\sum_{\alpha\in F_x} \varphi_\alpha(x)\Bigg) G(x) = G(x). \end{aligned}$$

□

As a simplified version of Proposition 33 we obtain (using Remark 11) the following basic result on C^1 extension, which clearly implies the basic part of Theorem 1 from Introduction.

Theorem 34. *Let X be a normed linear space and Y a Banach space such that the pair (X, Y) has properties (LE) and (LA$_1$). Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed. Suppose that $f : A \to Y$ satisfies condition (W) with G. Then f can be extended to a mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A.*

Similarly, from Proposition 32 we would obtain the following Lipschitz version with a rather large Lipschitz constant $16C^3L$, which clearly implies the moreover part of Theorem 1 from Introduction. However, using an idea that is implicitly contained in the monolithic proof of [JiS] we obtain a much better Lipschitz constant.

Theorem 35. *Let X be a normed linear space and Y a Banach space such that the pair (X, Y) has properties (LE) with $C_{\mathrm{E}} \geq 1$ and (LA$_1$) with $C_{\mathrm{A}} \geq 1$. Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed. Suppose that $f : A \to Y$ is L-Lipschitz and satisfies condition (W) with G such that $\sup_{x\in A}\|G(x)\| \leq L$. Let $\eta > 1$. Then f can be extended to an $\eta C_{\mathrm{A}} C_{\mathrm{E}} L$-Lipschitz mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A.*

In the proof we will utilise the following trick, which comes from the proof of [HJ1, Theorem 14].

Lemma 36. *Let X, Y be normed linear spaces, $\Omega \subset X$ open, and let $F, g : \Omega \to Y$ and $R > P \geq 0$ be such that F is P-Lipschitz, g is Fréchet differentiable with $\|Dg(x)\| \leq R$ for all $x \in \Omega$, and $\|F(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in \Omega$, where $\varepsilon(x) \leq (R - P)\operatorname{dist}(x, X \setminus \Omega)$. Then g is R-Lipschitz.*

Proof. Let $x, y \in \Omega$. If the line segment l with end points x and y lies in Ω, then $\|g(x) - g(y)\| \leq R\|x - y\|$ (see e.g. [HJ, Proposition 1.71]). Otherwise there is $z \in l \cap (X \setminus \Omega)$. Then

$$\begin{aligned} \|g(x) - g(y)\| &\leq \|g(x) - F(x)\| + \|F(x) - F(y)\| + \|F(y) - g(y)\| \leq \varepsilon(x) + P\|x - y\| + \varepsilon(y) \\ &\leq (R - P)\|x - z\| + P\|x - y\| + (R - P)\|y - z\| = R\|x - y\|. \end{aligned}$$

□

Proof of Theorem 35. Since the case $L = 0$ is trivial, we may suppose that $L > 0$. By Theorem 34 there is $F \in C^1(\Omega; Y)$ that is an extension of f and such that $DF = G$ on A. Put $\zeta = \frac{3}{4}\cdot 1 + \frac{1}{4}\eta < \frac{1+\eta}{2}$ and let $K > 1$ be such that $2\frac{\zeta L + L}{K-1} + L \leq \zeta L$. For $x \in A$ let $r_x > 0$ be such that $\|DF(y)\| \leq \zeta L$ for all $y \in U(x, Kr_x)$. Put $V = \bigcup_{x\in A} U(x, r_x)$, which is an open neighbourhood of A. We claim that $F\restriction_V$ is ζL-Lipschitz. Indeed, take $x, y \in V$. Let $u, v \in A$ be such that $x \in U(u, r_u)$, $y \in U(v, r_v)$. If $y \in U(u, Kr_u)$ or $x \in U(v, Kr_v)$, then $\|F(y) - F(x)\| \leq \zeta L\|y - x\|$. Otherwise $Kr_u \leq \|y - u\| \leq \|y - x\| + \|x - u\| < \|y - x\| + r_u$ and so $(K - 1)r_u < \|y - x\|$. Similarly, $(K - 1)r_v < \|y - x\|$. Hence

$$\begin{aligned} \|F(y) - F(x)\| &\leq \|F(y) - F(v)\| + \|F(v) - F(u)\| + \|F(u) - F(x)\| \leq \zeta L\|y - v\| + L\|v - u\| + \zeta L\|u - x\| \\ &< \zeta L r_v + L(r_v + \|y - x\| + r_u) + \zeta L r_u \leq \left(2\frac{\zeta L + L}{K - 1} + L\right)\|y - x\| \leq \zeta L\|y - x\|. \end{aligned}$$

Property (LE) implies that there is a $\zeta C_{\mathrm{E}} L$-Lipschitz extension $h : \Omega \to Y$ of $F\restriction_V$. In particular, $h\restriction_V$ is C^1-smooth and $h\restriction_A = f$. By the combination of Lemma 22 and [HJ, Lemma 7.49, (vi)⇒(i)] there is $\varphi \in C^1(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on A, and $\operatorname{supp}_{\mathrm{o}} \varphi \subset V$. (Note that $S := C^1(\Omega)$ is a locally determined partition ring. Indeed, since the local determination is obvious, by the easy remark preceding [HJ, Lemma 7.49] it is only necessary to show condition (iii) in the definition of the partition ring, which can be done analogously as it is proved in Lemma 22 for a different ring.) Put $\varepsilon = \frac{\eta-1}{2} C_{\mathrm{A}} C_{\mathrm{E}} L > 0$. By [HJ, Theorem 7.86] there is a $\frac{1+\eta}{2} C_{\mathrm{A}} C_{\mathrm{E}} L$-Lipschitz $H \in C^1(\Omega; Y)$ such that $\|h(x) - H(x)\| < \varepsilon(x)$ for $x \in \Omega$, where $\varepsilon(x) = \min\left\{\frac{\varepsilon}{\|D\varphi(x)\|+1}, \frac{\eta-1}{2} C_{\mathrm{A}} C_{\mathrm{E}} L \operatorname{dist}(x, X \setminus \Omega)\right\}$. Set

$$g(x) = \varphi(x) F(x) + \big(1 - \varphi(x)\big) H(x).$$

Clearly, $g \in C^1(\Omega; Y)$ and $g = f$ on A. Further, $Dg(x) = \varphi(x)DF(x) + D\varphi(x)\cdot F(x) + \big(1-\varphi(x)\big)DH(x) - D\varphi(x)\cdot H(x)$ for $x \in \Omega$. Note that $D\varphi(x) = 0$ for $x \in A$, resp. $x \in \Omega \setminus V$, since φ attains its maximum, resp. minimum, at x. In particular, $Dg(x) = DF(x) = G(x)$ for $x \in A$ and $\|Dg(x)\| = \|DH(x)\| \leq \eta C_{\mathrm{A}} C_{\mathrm{E}} L$ for $x \in \Omega \setminus V$. Also,

$$\begin{aligned}\|Dg(x)\| &\leq \varphi(x)\|DF(x)\| + \big(1-\varphi(x)\big)\|DH(x)\| + \|D\varphi(x)\|\|F(x) - H(x)\| \\ &\leq \frac{1+\eta}{2}L\varphi(x) + \frac{1+\eta}{2}C_{\mathrm{A}}C_{\mathrm{E}}L\big(1-\varphi(x)\big) + \|D\varphi(x)\|\frac{\varepsilon}{\|D\varphi(x)\|+1} \leq \eta C_{\mathrm{A}}C_{\mathrm{E}}L\end{aligned}$$

for $x \in V$. Notice that $\|h(x)-g(x)\| = \|h(x)-H(x)\| < \varepsilon(x)$ for $x \in \Omega \setminus V$ and $\|h(x)-g(x)\| = \big(1-\varphi(x)\big)\|h(x)-H(x)\| \leq \varepsilon(x)$ for $x \in V$. Hence $\|h(x) - g(x)\| \leq \big(\eta C_{\mathrm{A}}C_{\mathrm{E}}L - \frac{1+\eta}{2}C_{\mathrm{A}}C_{\mathrm{E}}L\big)\operatorname{dist}(x, X \setminus \Omega)$ for $x \in \Omega$. Since h is $\frac{1+\eta}{2}C_{\mathrm{A}}C_{\mathrm{E}}L$-Lipschitz, Lemma 36 implies that g is $\eta C_{\mathrm{A}}C_{\mathrm{E}}L$-Lipschitz.

□

Note an interesting feature of the above proof: it shows that if we are able to extend a Lipschitz mapping from A to a C^1-smooth mapping defined on a neighbourhood of A, then using the already known results from [HJ1] we can extend it to a Lipschitz C^1-smooth mapping defined on the whole space such that the Lipschitz constant of the extension is almost optimal. So, curiously, the whole proof of Theorem 35 goes as follows: first we prove the second (Lipschitz) part of Theorem 1 with some large constant K_1, then we use it to prove the first part of Theorem 1, and then we use the first part to prove the second part with an almost optimal constant K_2.

Remark 37. We remark that in [JZ, Section 6] there is a version of Theorem 35 with a Lipschitz constant of the extension that is worse, but no more than 4 times than here. Note that the constant obtained in [JiS] looks optically the same as in Theorem 35 ($\eta C C_{\mathrm{E}}$), but whereas our C_{A} is the constant of the approximation on balls, the constant C from [JiS] is the constant of the approximation on the whole space. As was remarked in Remark 18, in general "$C \leq C_{\mathrm{A}} \leq 2C$" (for the "optimal constants"), but in some cases "$C = C_{\mathrm{A}}$". However there may be spaces X, Y for which "$C < C_{\mathrm{A}}$", so it is not impossible that the results in [JiS] give for some X, Y a better Lipschitz constant in the case $\Omega = X$. (Note that in [JiS] the extension is considered only from closed sets to the whole space.)

4. Higher order smoothness on the complement

As we already mentioned, H. Whitney in his extension theorem from [Wh1] actually constructed a function $g \in C^1(\mathbb{R}^n)$ that extends $f : A \to \mathbb{R}$ and is analytic on $\mathbb{R}^n \setminus A$. In this section we show that under certain natural assumptions a similar "exterior regularity" holds also in the infinite-dimensional case, namely we can construct the extending mapping $g \in C^1(X; Y)$ so that it is C^k-smooth on $X \setminus A$ (for some $k \in \mathbb{N} \cup \{\infty\}$, $k > 1$).

Lemma 38. *Let X, Y be normed linear spaces, $\Omega \subset X$ open, $A \subset \Omega$ relatively closed, and $F \in C^1(\Omega; Y)$. Let $h \in C^1(\Omega \setminus A; Y)$ be such that $\|F(x) - h(x)\| \leq \varepsilon(x)$ and $\|DF(x) - Dh(x)\| \leq \varepsilon(x)$ for all $x \in \Omega \setminus A$, where $\varepsilon(x) \leq \operatorname{dist}^2(x, A)$. Set $g = F$ on A and $g = h$ on $\Omega \setminus A$. Then $g \in C^1(\Omega; Y)$ and $Dg = DF$ on A.*

Proof. Set $G = g - F$. Clearly, $DG(x) = Dh(x) - DF(x)$ for every $x \in \Omega \setminus A$. Let $x \in A$. We claim that $DG(x) = 0$. Note that $\|G(y)\| \leq \operatorname{dist}^2(y, A)$ for every $y \in \Omega$. So for any $v \in X$ such that $x + v \in \Omega$ we get

$$\|G(x+v) - G(x) - 0\| = \|G(x+v)\| \leq \operatorname{dist}^2(x+v, A) \leq \|x+v-x\|^2 = \|v\|^2 = o(\|v\|), \quad v \to 0.$$

Further, the continuity of DG on $\Omega \setminus A$ is clear. The continuity of DG at any $x \in A$ follows from the fact that $\|DG(y) - DG(x)\| = \|DG(y)\| \leq \varepsilon(y) \leq \operatorname{dist}^2(y, A) \leq \|y - x\|^2$ whenever $y \in \Omega \setminus A$. Since $g = G + F$, it follows that $g \in C^1(\Omega; Y)$ and $Dg = DF$ on A.

□

Using Lemmata 38 and 36, under assumption (LA_k) we easily obtain improved versions of Proposition 33 and 32 with higher order smoothness on the complement of A.

Proposition 39. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has property (LA_k). Let $\Omega \subset X$ be open and let $A \subset \Omega$ be relatively closed such that the pair (A, Y) has property (LLE). Suppose that $f : A \to Y$ satisfies condition (W) with G. Then f can be extended to a mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A and g is C^k-smooth on $\Omega \setminus A$.*

Proof. By Proposition 33 there is $F \in C^1(\Omega; Y)$ that is an extension of f and such that $DF = G$ on A. By [HJ, Theorem 7.95, (i)⇒(iii)] there is $h \in C^k(\Omega \setminus A; Y)$ such that $\|F(x) - h(x)\| < \varepsilon(x)$ and $\|DF(x) - Dh(x)\| < \varepsilon(x)$ for all $x \in \Omega \setminus A$, where $\varepsilon(x) = \operatorname{dist}^2(x, A)$. Set $g = F = f$ on A and $g = h$ on $\Omega \setminus A$. Then g is C^k-smooth on $\Omega \setminus A$, and $g \in C^1(\Omega; Y)$ and $Dg = DF = G$ on A by Lemma 38.

□

Proposition 40. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has property (LA_1) with $C \geq 1$ and moreover it has property (LA_k). Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed such that each component of Ω has a non-empty intersection with A. Suppose that*

(a) every Q-Lipschitz mapping from A to Y can be extended to a CQ-Lipschitz mapping on Ω and
(b) the pair (A, Y) has property (LLE).

Let $L \geq 0$ and let $f\colon A \to Y$ be an L-Lipschitz mapping that satisfies condition (W) with G such that $\sup_{x\in A}\|G(x)\| \leq L$. Then f can be extended to a $17C^3L$-Lipschitz mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A and g is C^k-smooth on $\Omega \setminus A$.

Proof. Since the case $L = 0$ is trivial, we may suppose that $L > 0$. Put $P = 16C^3L$ and $R = 17C^3L$. By Proposition 32 there exists a P-Lipschitz mapping $F \in C^1(\Omega; Y)$ that is an extension of f and such that $DF = G$ on A. Let $\varepsilon(x) = \min\{\operatorname{dist}^2(x, A), (R - P)\operatorname{dist}(x, X \setminus \Omega), R - P\}$ for $x \in \Omega$ and note that ε is continuous and $\varepsilon(x) > 0$ for $x \in \Omega \setminus A$. By [HJ, Theorem 7.95, (i)⇒(iii)] there is $h \in C^k(\Omega \setminus A; Y)$ such that $\|F(x) - h(x)\| < \varepsilon(x)$ and $\|DF(x) - Dh(x)\| < \varepsilon(x)$ for all $x \in \Omega \setminus A$. Set $g = F = f$ on A and $g = h$ on $\Omega \setminus A$. Then g is C^k-smooth on $\Omega \setminus A$, and $g \in C^1(\Omega; Y)$ and $Dg = DF = G$ on A by Lemma 38.

Further, $\|Dg(x)\| = \|G(x)\| \leq L \leq R$ for every $x \in A$ and $\|Dg(x)\| \leq \|DF(x)\| + \|DF(x) - Dg(x)\| = \|DF(x)\| + \|DF(x) - Dh(x)\| < P + \varepsilon(x) \leq P + R - P = R$ for $x \in \Omega \setminus A$. Also, F is P-Lipschitz and $\|F(x) - g(x)\| \leq \varepsilon(x) \leq (R - P)\operatorname{dist}(x, X \setminus \Omega)$ for $x \in \Omega$. So Lemma 36 implies that g is R-Lipschitz.

□

As a consequence of Proposition 39 we immediately obtain (using Remark 11) our main result for C^1 extensions.

Theorem 41. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has properties (LE) and (LA$_k$). Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed. Suppose that $f\colon A \to Y$ satisfies condition (W) with G. Then f can be extended to a mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A and g is C^k-smooth on $\Omega \setminus A$.*

The proof of the following theorem is essentially the same as that of Proposition 40 except for using Theorem 35 instead of Proposition 32.

Theorem 42. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has properties (LE) with $C_{\mathrm{E}} \geq 1$ and (LA$_1$) with $C_{\mathrm{A}} \geq 1$, and moreover it has property (LA$_k$). Let $\Omega \subset X$ be an open set and let $A \subset \Omega$ be relatively closed. Suppose that $f\colon A \to Y$ is L-Lipschitz and satisfies condition (W) with G such that $\sup_{x\in A}\|G(x)\| \leq L$. Let $\eta > 1$. Then f can be extended to an $\eta C_{\mathrm{A}}C_{\mathrm{E}}L$-Lipschitz mapping $g \in C^1(\Omega; Y)$ such that $Dg = G$ on A and g is C^k-smooth on $\Omega \setminus A$.*

Proof. Put $P = \frac{\eta+1}{2}C_{\mathrm{A}}C_{\mathrm{E}}L$ and $R = \eta C_{\mathrm{A}}C_{\mathrm{E}}L$. By Theorem 35 there is a P-Lipschitz $F \in C^1(\Omega; Y)$ that is an extension of f and such that $DF = G$ on A. The rest of the proof is a word for word copy of the proof of Proposition 40.

□

Example 43. Combining the facts from Examples 9 and 21 we conclude that the pair (X, Y) has both properties (LE) and (LA$_k$) in particular in the following cases:

(a) X is finite dimensional, Y is an arbitrary Banach space, and $k = \infty$.
(b) There is a bi-Lipschitz homeomorphism $\Phi\colon X \to c_0(\Gamma)$ into with C^k-smooth component functions, Y is an absolute Lipschitz retract.
(c) X is separable and admits a C^k-smooth Lipschitz bump (in particular, X^* is separable and $k = 1$), Y is an absolute Lipschitz retract.
(d) X is a subspace of $L_p(\mu)$ for some measure μ and $1 < p < \infty$ (resp. of some super-reflexive Banach lattice with a (long) unconditional basis or a weak unit) with $\operatorname{dens} X < \omega_\omega$, Y is an absolute Lipschitz retract, and $k = 1$.
(e) X is super-reflexive, Y is finite-dimensional, and $k = 1$.
(f) X is a Banach space with an unconditional Schauder basis that has an equivalent norm with modulus of smoothness of power type 2 and admits a C^k-smooth Lipschitz bump, Y is a Banach space that has an equivalent norm with modulus of convexity of power type 2.

Recall that if a space admits an equivalent C^k-smooth norm, then it also admits a C^k-smooth Lipschitz bump (just compose the norm with a function from $C^\infty(\mathbb{R})$ with support in $[1, 2]$).

Using the above we obtain that the following pairs (X, Y) of classical spaces have both properties (LE) and (LA$_k$) (the space $C_{\mathrm{ub}}(P)$ is as in Example 9):

<table>
<tr><th colspan="2">X</th><th>Y</th><th>k</th><th>follows from</th></tr>
<tr><td rowspan="4">$L_p(\mu)$</td><td>$1 < p < \infty$</td><td>finite-dimensional</td><td rowspan="2">1</td><td>(e)</td></tr>
<tr><td>$1 < p < \infty$, $\operatorname{dens} X < \omega_\omega$</td><td rowspan="2">$c_0(\Gamma)$, $\ell_\infty(\Gamma)$, $C_{\mathrm{ub}}(P)$</td><td>(d)</td></tr>
<tr><td>$1 < p < \infty$, separable</td><td rowspan="2">∞ for $p \in 2\mathbb{N}$,
$\lceil p \rceil - 1$ for $p \notin 2\mathbb{N}$</td><td>(c), [HJ, Th. 5.106]</td></tr>
<tr><td>$2 \leq p < \infty$, separable</td><td>$L_q(\nu)$, $1 < q \leq 2$</td><td>(f), [La, Cor. p. 128], [AK, Th. 6.1.6], [DGZ, Cor. V.1.2], [HJ, Th. 5.106]</td></tr>
<tr><td colspan="2">$c_0(\Gamma)$</td><td rowspan="2">$c_0(\Gamma)$, $\ell_\infty(\Gamma)$, $C_{\mathrm{ub}}(P)$</td><td rowspan="2">∞</td><td>(b)</td></tr>
<tr><td colspan="2">$C([0, \alpha])$, α countable ordinal</td><td>(c), [HJ, Th. 5.127]</td></tr>
</table>

Some of our examples are more general (namely (f) and the fourth line in the table) than those in [JS2] and (d), (e) (and the first two lines in the table) are completely new.

Another consequence of Propositions 39 and 40 is the following result on extensions from special subsets of X in which we do not assume property (LE).

Corollary 44. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has property (LA$_k$). Suppose that $A \subset X$ is one of the following types:*

(a) A is an image of a closed convex bounded set with a non-empty interior under a bi-Lipschitz automorphism of X;
(b) A is a Lipschitz submanifold of X;
(c) A is the closure of a Lipschitz domain in X.

Suppose that $f: A \to Y$ satisfies condition (W) with G. Then f can be extended to a mapping $g \in C^1(X; Y)$ such that $Dg = G$ on A and g is C^k-smooth on $X \setminus A$.

Moreover, if (a) holds, f is Lipschitz, and the mapping G is bounded, then we can additionally assert that g is Lipschitz.

Proof. The basic part directly follows from Proposition 39 and Lemma 15. If (a) holds, then A is a Lipschitz retract of X by Lemma 5. So, using Fact 4, we can apply Proposition 40 with some sufficiently large $C \geq 1$.

□

Remark 45. We can apply Corollary 44 e.g. in the case when $X = L_p(\mu)$ separable, $1 < p < \infty$, and Y is an arbitrary Banach space with $k = \infty$ for p even integer, $k = \lceil p \rceil - 1$ otherwise (Example 21(b) with [La, Cor. p .128], [AK, Theorem 6.1.6] and [HJ, Theorem 5.106]). In the case that $Y = L_q(\nu)$ for $q > p$ and X, Y are infinite-dimensional, then the pair (X, Y) does not have property (LE), see Example 9, and so it is not possible to apply Theorems 41, 42.

5. Extension from open sets

Following Whitney's article [Wh2] we will apply the results of preceding sections to obtain results on extension of C^1-smooth mappings from quasiconvex open sets. In fact we will work with more general open sets (which are "weakly quasiconvex", see Definition 48).

The following notion of a quasiconvex space is now a standard tool in Geometric Analysis. For the (standard) definition and properties of the length (variation) of a curve in a metric space see e.g. [Ch].

Definition 46. We say that a metric space (X, ρ) is c-*quasiconvex* (where $c \geq 1$) if for each $x, y \in X$ there exists a continuous rectifiable curve $\gamma: [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{len} \gamma \leq c\rho(x, y)$, where $\operatorname{len} \gamma$ is the length of the curve γ. We say that X is *quasiconvex* if it is c-quasiconvex for some $c \geq 1$.

Note that convex subsets of normed linear spaces are 1-quasiconvex.

Remark 47. It is well-known and easy to prove that each bi-Lipschitz image of a quasiconvex metric space is quasiconvex.

Definition 48. We say that a subset U of a metric space (X, ρ) has property (WQ) if for each $a \in \partial U$ there exist $r > 0$ and $c \geq 1$ such that for each $x, y \in U \cap U(a, r)$ there exists a continuous rectifiable curve $\gamma: [0, 1] \to U$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{len} \gamma \leq c\rho(x, y)$.

Note that each quasiconvex subset of a metric space clearly has property (WQ), i.e. it is "weakly quasiconvex".

Proposition 49. *Let X be a normed linear space and Y a Banach space such that the pair (X, Y) has property (LA$_1$). Let $U \subset X$ be an open set with property (WQ) such that $(\overline{U}, Y)$ has property (LLE). Let $f \in C^1(U; Y)$. Then f can be extended to an $F \in C^1(X; Y)$ if and only if the mapping $Df: U \to \mathcal{L}(X; Y)$ has a continuous extension $G: \overline{U} \to \mathcal{L}(X; Y)$.*

Proof. $\Rightarrow$ is obvious.

$\Leftarrow$ First we prove the following claim:

($*$) For each $a \in \partial U$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(y) - f(x) - G(a)[y - x]\| \leq \varepsilon \|y - x\|$ whenever $x, y \in U(a, \delta) \cap U$.

So fix $a \in \partial U$ and let $\varepsilon > 0$. Let $r > 0$ and $c \geq 1$ be from property (WQ). By the continuity of G there is $0 < \delta \leq r$ such that

$$\|Df(z) - G(a)\| < \frac{\varepsilon}{2c} \text{ whenever } z \in U(a, 3c\delta) \cap U. \tag{15}$$

Now let $x, y \in U(a, \delta) \cap U$. Since $\delta \leq r$, we can choose a continuous rectifiable curve $\gamma: [0, 1] \to U$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{len} \gamma \leq c\|x - y\|$. It is easy to check that $\langle \gamma \rangle := \gamma([0, 1]) \subset U(a, 3c\delta) \cap U$. By Remark 24(c) for each $z \in U$ there is $\delta_z > 0$ such that

$$\|f(v) - f(u) - Df(z)[v - u]\| \leq \frac{\varepsilon}{2c}\|v - u\| \quad \text{whenever } u, v \in U(z, \delta_z). \tag{16}$$

Let $\lambda > 0$ be a Lebesgue number (see [E, p. 276]) of the covering $\{U(z, \delta_z);\ z \in \langle \gamma \rangle\}$ of the compact set $\langle \gamma \rangle$. By the uniform continuity of γ we choose $\Delta > 0$ such that $\|\gamma(t) - \gamma(s)\| < \lambda$ whenever $0 \leq s \leq t \leq 1$ and $t - s < \Delta$. Further, choose points

$0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ such that $t_i - t_{i-1} < \Delta$ for $i = 1, \dots, n$ and denote $x_i = \gamma(t_i)$. Then clearly $x_0 = x$, $x_n = y$, and

$$\sum_{i=1}^{n} \|x_i - x_{i-1}\| \le \operatorname{len} \gamma \le c\|y - x\|. \tag{17}$$

The choice of λ, Δ, and $t_0, \dots, t_n$ implies that for each $1 \le i \le n$ there exists a point $z_i \in \langle\gamma\rangle$ such that $x_{i-1}, x_i \in U(z_i, \delta_{z_i})$ and consequently by (16)

$$\big\| f(x_i) - f(x_{i-1}) - Df(z_i)[x_i - x_{i-1}] \big\| \le \frac{\varepsilon}{2c} \|x_i - x_{i-1}\|. \tag{18}$$

Using (18), (15), and (17) we obtain that

$$\begin{aligned} \big\| f(y) - f(x) - G(a)[y - x] \big\| &= \left\| \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) - \sum_{i=1}^{n} G(a)[x_i - x_{i-1}] \right\| \\ &\le \left\| \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1}) - Df(z_i)[x_i - x_{i-1}]\big) \right\| + \left\| \sum_{i=1}^{n} \big(Df(z_i) - G(a)\big)[x_i - x_{i-1}] \right\| \\ &\le \frac{\varepsilon}{2c} \sum_{i=1}^{n} \|x_i - x_{i-1}\| + \frac{\varepsilon}{2c} \sum_{i=1}^{n} \|x_i - x_{i-1}\| \le \varepsilon \|y - x\|, \end{aligned}$$

and claim $(*)$ is proved.

To finish the proof, by Proposition 33 it is sufficient to define an extension $\tilde{f} \colon \overline{U} \to Y$ of f that satisfies condition (W) with G. Set $\tilde{f}(a) = f(a)$ for $a \in U$. Now suppose that $a \in \partial U$. Choose $\delta > 0$ corresponding to $\varepsilon = 1$ in claim $(*)$. Then $\|f(y) - f(x)\| \le \big(\|G(a)\| + 1\big)\|y - x\|$ whenever $x, y \in U(a, \delta) \cap U$. It follows that there exists $\tilde{f}(a) = \lim_{x \to a, x \in U} f(x)$.

To show that $\tilde{f}$ satisfies condition (W) with G consider an arbitrary $a \in \overline{U}$. We will show that $G(a)$ is a strict derivative of $\tilde{f}$ at a with respect to $\overline{U}$. Since the case $a \in U$ is obvious by Remark 24(c) we suppose that $a \in \partial U$. Let $\varepsilon > 0$. Choose $\delta > 0$ corresponding to this ε in claim $(*)$. If $x, y \in U(a, \delta) \cap \overline{U}$ are arbitrary, then we can choose sequences $\{x_n\}$, $\{y_n\}$ of points from $U(a, \delta) \cap U$ such that $x_n \to x$ and $y_n \to y$. Then

$$\big\| \tilde{f}(y) - \tilde{f}(x) - G(a)[y - x] \big\| = \lim_{n \to \infty} \big\| f(y_n) - f(x_n) - G(a)[y_n - x_n] \big\| \le \lim_{n \to \infty} \varepsilon \|y_n - x_n\| = \varepsilon \|y - x\|.$$

This completes the proof.

□

Remark 50. The assumption "Df has a continuous extension $G \colon \overline{U} \to \mathcal{L}(X; Y)$" in the above proposition is equivalent to the property "$\lim_{x \to a, x \in U} Df(x)$ exists for each $a \in \partial U$". This follows easily e.g. from [E, Lemma 4.3.16].

Theorem 51. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has properties (LE) and (LA$_k$). Let $U \subset X$ be an open set with property (WQ), $f \in C^1(U; Y)$, and suppose that the mapping $Df \colon U \to \mathcal{L}(X; Y)$ has a continuous extension $G \colon \overline{U} \to \mathcal{L}(X; Y)$. Then f can be extended to a mapping $g \in C^1(X; Y)$ such that g is C^k-smooth on $X \setminus \overline{U}$.*

Proof. The pair $(\overline{U}, Y)$ has property (LLE) by Remark 11, so f can be extended to a mapping $F \in C^1(X; Y)$ by Proposition 49. Since F satisfies condition (W) on $\overline{U}$ by Fact 26, the existence of the desired g follows from Theorem 41 (applied to the mapping $F\restriction_{\overline{U}}$ and $\Omega = X$).

□

Theorem 52. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has properties (LE) with $C_{\mathrm{E}} \ge 1$ and (LA$_1$) with $C_{\mathrm{A}} \ge 1$, and moreover it has property (LA$_k$). Let $U \subset X$ be an open set with property (WQ), let $f \in C^1(U; Y)$ be L-Lipschitz, and suppose that the mapping $Df \colon U \to \mathcal{L}(X; Y)$ has a continuous extension $G \colon \overline{U} \to \mathcal{L}(X; Y)$. Let $\eta > 1$. Then f can be extended to an $\eta C_{\mathrm{A}} C_{\mathrm{E}} L$-Lipschitz mapping $g \in C^1(X; Y)$ such that g is C^k-smooth on $X \setminus \overline{U}$.*

Proof. The pair $(\overline{U}, Y)$ has property (LLE) by Remark 11, so f can be extended to a mapping $F \in C^1(X; Y)$ by Proposition 49. Clearly $DF = G$ on $\overline{U}$ and so F satisfies condition (W) with G on $\overline{U}$ by Fact 26. Since $\|Df(x)\| \le L$ for each $x \in U$, it follows that $\|G(x)\| \le L$ for each $x \in \overline{U}$. So the existence of the desired g follows from Theorem 42 (applied to the mapping $F\restriction_{\overline{U}}$ and $\Omega = X$).

□

For some more special open sets with property (WQ) we do not need property (LE) (cf. Remark 45).

Corollary 53. *Let X be a normed linear space, Y a Banach space, and $k \in \mathbb{N} \cup \{\infty\}$ such that the pair (X, Y) has property (LA$_k$). Suppose that $U \subset X$ is one of the following types:*

(a) U is an image of an open convex bounded set under a bi-Lipschitz automorphism of X;
(b) U is a Lipschitz domain in X.

Let $f \in C^1(U;Y)$ and suppose that the mapping $Df\colon U \to \mathcal{L}(X;Y)$ has a continuous extension $G\colon \overline{U} \to \mathcal{L}(X;Y)$. Then f can be extended to a mapping $g \in C^1(X;Y)$ such that g is C^k-smooth on $X \setminus \overline{U}$.

Moreover, if (a) holds and f is Lipschitz, then we can additionally assert that g is Lipschitz.

Proof. We claim that U has property (WQ). In case (a) the claim holds by Remark 47. If (b) holds, choose an arbitrary $a \in \partial U$ and then V, E, and Φ as in Definition 7. Since $\Phi(U \cap V)$ is convex, we obtain that $U \cap V$ is c-quasiconvex for some $c \geq 1$ by Remark 47. Now choose $r > 0$ such that $U(a,r) \subset V$ and consider arbitrary $x, y \in U \cap U(a,r) \subset U \cap V$. Then there exists a continuous rectifiable curve $\gamma\colon [0,1] \to U \cap V \subset U$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{len}\gamma \leq c\|y - x\|$. Thus we have proved that U has property (WQ).

Now observe that the pair $(\overline{U}, Y)$ has property (LLE) by Lemma 15 (with assumption (a) or (c)). So f can be extended to a mapping $F \in C^1(X;Y)$ by Proposition 49. Clearly $DF = G$ on $\overline{U}$ and so F satisfies condition (W) with G on $\overline{U}$ (Fact 26). Moreover, if (a) holds and f is L-Lipschitz, then its (continuous) extension F is L-Lipschitz on $\overline{U}$ and $\|G(x)\| = \|DF(x)\| \leq L$ for each $x \in \overline{U}$. The existence of the desired g now follows from Corollary 44 (applied with $A := \overline{U}$ and $f := F\restriction_{\overline{U}}$). □

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