(a) $L^p(\mu; X)$ is a Banach space

- it is a normed linear space by the definition and Minkowski inequality

completeness:

- $p = \infty$: Let $(f_n)_{n=1}^{\infty}$ be a $\| \ \|_\infty$ - Cauchy

Then $\forall k \in \mathbb{N} \ \exists n_0 = n_0(k) \ \forall m, n \geq n_0$:
$$\| f_n - f_m \|_\infty < \frac{1}{k}$$

So, $N_{m,n,k} = \{ \omega \ ; \ \| f_n(\omega) - f_m(\omega) \| \geq \frac{1}{k} \}$

has measure 0 whenever $m, n \geq n_0(k)$

$$N := \bigcup_{k \in \mathbb{N}} \ \bigcup_{m,n \geq n_0(k)} N_{m,n,k} \Rightarrow N \text{ has measure } 0$$

and (f_n) is uniformly Cauchy on $\Omega \setminus N$.

Since X is complete, it follows that f_n is pointwise convergent on $\Omega \setminus N$. Being moreover uniformly Cauchy, it is uniformly convergent

$\big[f(\omega) := \lim f_n(\omega) \ , \ \omega \in \Omega \setminus N$

$\varepsilon > 0 \Rightarrow \exists n_0 \ \forall m, n \geq n_0 \ \| f_m - f_n \|_\infty < \varepsilon$

$n \geq n_0 . \ \forall \omega \in \Omega \setminus N \ \forall m \geq n_0 \ \| f_n(\omega) - f_m(\omega) \| < \varepsilon$

$$\Downarrow$$

$$\| f_n(\omega) - f(\omega) \|$$

$\Rightarrow \| f_n(\omega) - f(\omega) \| \leq \varepsilon \ \big]$

• $p = \infty$ (H.W. $p \in [1, \infty)$)

Suppose $(f_n) \subset L^p(\mu; X)$, $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$

Define $g_n(\omega) := \|f_n(\omega)\|$, $\omega \in \Omega$

By the definition $g_n \in L^p(\mu)$ & $\|g_n\|_p = \|f_n\|_p$

So, $\sum_{n=1}^{\infty} \|g_n\|_p < \infty$. Since $L^p(\mu)$ is complete,

we get that $\sum_{n=1}^{\infty} g_n$ converges in $L^p(\mu)$.

$g := \sum_{n=1}^{\infty} g_n \in L^p(\mu)$ (convergence in $L^p(\mu)$)

Further, there is a subsequence of the sequence of partial sums converging a.e. . But $g_n \geq 0$, so $g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$ a.e.

Hence, for almost all $\omega \in \Omega$ we have

$\sum_{n=1}^{\infty} g_n(\omega) < \infty \Rightarrow \sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty \Rightarrow \sum_{n=1}^{\infty} f_n(\omega)$ converges in X

Set $f(\omega) = \sum_{n=1}^{\infty} f_n(\omega)$ ~~μa~~ $\Rightarrow$ f is defined a.e.

Moreover, f is strongly μ-measurable (Lemma 4)

$\|f(\omega)\| \leq \sum_{n=1}^{\infty} \|f_n(\omega)\| = g(\omega)$ a.e. $\Rightarrow f \in L^p(\mu; X)$

Finally, $\|f(\omega) - \sum_{k=1}^{n} f_k(\omega)\| = \|\sum_{k>n} f_k(\omega)\| \leq \sum_{k>n} \|f_k(\omega)\|$ a.e.

$\Rightarrow \|f - \sum_{k=1}^{n} f_k(\omega)\|_p \leq \|\omega \mapsto \sum_{k>n} \|f_k(\omega)\|\|_p \leq$

$\leq \sum_{k>n} \|\omega \mapsto \|f_k(\omega)\|\|_p = \sum_{k>n} \|f_k\|_p \to 0$

So, $\sum_{n=1}^{\infty} f_n = f$ in $L^p(\mu; X)$, this completes the proof of completeness.

(b) $L^1(\mu; X)$ = Bochner-integrable functions

[By definitions and Theorem 8]

(c) X Hilbertspace $\Rightarrow$ $L^2(\mu; X)$ is a Hilbertspace

Γ $\langle f, g\rangle := \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega)$

• $\omega \mapsto \langle f(\omega), g(\omega)\rangle$ is measurable

Γ $f(\omega) =$ a.e. $-\lim u_n(\omega)$
$g(\omega) =$ a.e. $-\lim v_n(\omega)$ $\quad u_n, v_n$ simple measurable

$\omega \mapsto \langle u_n(\omega), v_n(\omega)\rangle$ is simple measurable
$\langle f(\omega), g(\omega)\rangle = \lim \langle u_n(\omega), v_n(\omega)\rangle$ a.e. $\rfloor\rfloor$

• $\omega \mapsto \langle f(\omega), g(\omega)\rangle$ is integrable

Γ $\int_\Omega |\langle f(\omega), g(\omega)\rangle| \, d\mu(\omega) \leq \int_\Omega \|f(\omega)\| \cdot \|g(\omega)\| \, d\mu(\omega)$

$\leq \left(\int_\Omega \|f(\omega)\|^2 d\mu(\omega)\right)^{1/2} \left(\int_\Omega \|g(\omega)\|^2 d\mu(\omega)\right)^{1/2} = \|f\|_2 \|g\|_2$ $\rfloor\rfloor$

Hence, $\langle f, g\rangle$ is well-defined. Clearly it is linear in f and $\overline{\langle f, g\rangle} = \langle g, f\rangle$.

Finally, $\langle f, f\rangle = \int_\Omega \langle f(\omega), f(\omega)\rangle d\mu(\omega) = \int_\Omega \|f(\omega)\|^2 d\mu(\omega)$

$= \|f\|_2^2$. So, it is an inner product generating the norm. $\rfloor$