

A unital Banach algebra

(1) $G(A)$ is an open subset of A

By Lemma 6(5) : $x \in G(A) \Rightarrow U(x, \frac{1}{\|x^{-1}\|}) \subset A$

(2) $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$

Let $\varphi(x) := x^{-1}$. Then $\varphi(\varphi(x)) = x$, so φ is a bijection $G(A)$ onto $G(A)$ and $\varphi^{-1} = \varphi$.

So, it is enough to show that φ is cts:

Suppose $x_n \rightarrow x$ in $G(A)$. There is no s.c. for $n \geq n_0$ we have $\|x_n - x\| < \frac{1}{\|x^{-1}\|}$

Then, by Lemma 6(5), we have for $n \geq n_0$

$$\|x_n^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \|x_n - x\|}{1 - \|x^{-1}\| \|x_n - x\|} \xrightarrow{n \rightarrow \infty} 0$$

Thus $x_n^{-1} \rightarrow x^{-1}$.

(3) $(x_n) \subset G(A)$, $x_n \rightarrow x \in A \setminus G(A) \Rightarrow \|x_n^{-1}\| \rightarrow \infty$

Since $x_n \in G(A)$, $x \notin G(A)$, by Lemma 6(5)

we have $\|x_n - x\| \geq \frac{1}{\|x_n^{-1}\|}$. Since $\|x_n - x\| \rightarrow 0$,

we deduce $\frac{1}{\|x_n^{-1}\|} \rightarrow 0$, so $\|x_n^{-1}\| \rightarrow \infty$