

A Let A be a unital B -algebra, $e \in A$ be the unit
 $h \in \Delta(A)$, i.e. $h: A \rightarrow \mathbb{C}$ is a nonzero homomorphism.
 Then

① $h(e) = 1$

Γ $h(e) = h(e \cdot e) = h(e) \cdot h(e) \Rightarrow h(e) = 1 \text{ or } h(e) = 0$

If $h(e) = 0$, then $\forall x \in A: h(x) = h(e \cdot x) = h(e) \cdot h(x) = 0 \cdot h(x) = 0$

② $x \in G(A) \Rightarrow h(x) \neq 0$

Γ $1 = h(e) = h(x \cdot x^{-1}) = h(x) \cdot h(x^{-1})$

③ $\ker h$ is a maximal ideal

Γ As h is a nonzero homomorphism, $\ker h$ is an ideal.

As h is linear, $\ker h$ is of codimension 1, so it is maximal.

④ $\forall x \in A: h(x) \in \sigma(x)$

Γ $h(h(x)e - x) = h(x)h(e) - h(x) \stackrel{①}{=} 0$

$\stackrel{②}{\Rightarrow} h(x)e - x \notin G(A) \Rightarrow h(x) \in \sigma(x)$

⑤ $\|h\| = 1$

Γ B_f ④ $|h(x)| \leq r(x) \leq \|x\| \Rightarrow \|h\| \leq 1$

R_f ① $h(e) = 1 \Rightarrow \|h\| \geq 1$

B Let A be a Banach algebra (unital or not) and $h \in \Delta(A)$.

① $\exists! \tilde{h} \in \Delta(A^+)$ s.t. $\tilde{h}(a, 0) = h(a)$ for $a \in A$

uniqueness: by **A1** necessarily $\tilde{h}(0, 1) = 1$,
so $\tilde{h}(a, \lambda) = h(a) + \lambda$ for $(a, \lambda) \in A^+$

Existence: Define $\tilde{h}(a, \lambda) = h(a) + \lambda$, $(a, \lambda) \in A^+$

• Clearly \tilde{h} is linear

$$\begin{aligned} \tilde{h}((a, \lambda)(b, \mu)) &= \tilde{h}(ab + \lambda b + \mu a + \lambda \mu) = \\ &= h(ab + \lambda b + \mu a + \lambda \mu) = h(a)h(b) + \lambda h(b) + \mu h(a) + \lambda \mu \\ &= (h(a) + \lambda)(h(b) + \mu) = \tilde{h}(a, \lambda) \tilde{h}(b, \mu) \end{aligned}$$

• $\tilde{h} \neq 0$ as $\tilde{h}(0, 1) = 1 \neq 0$

② $\|h\| \leq 1$

Consider \tilde{h} given by ①. by **A5** we have $\|\tilde{h}\| = 1$.
so, $\|h\| \leq 1$ ($h(a) = \tilde{h}(a, 0)$)

③ $h(x) \in \sigma(x)$ for $x \in A$

A unital ... by **A5**

A not unital $\Rightarrow h(x) = \tilde{h}(x, 0) \in \sigma_{A^+}^{\tilde{h}}(x, 0) = \sigma_A^h(x)$