(a) Let A be a Banach algebra with involution. Then A^+ is again a B-algebra with involution if we set
$(a,\lambda)^* = (a^*, \bar{\lambda})$, $(a,\lambda) \in A^+$

⌈ The only nontrivial axiom is $((a,\lambda)(b,\mu))^* = (b,\mu)^*(a,\lambda)^*$

And this holds:

$((a,\lambda)(b,\mu))^* = (ab + \lambda b + \mu a, \lambda\mu)^* = ((ab+\lambda b+\mu a)^*, \bar{\lambda}\bar{\mu}) =$
$= (b^*a^* + \bar{\lambda} b^* + \bar{\mu}\bar{a}, \bar{\lambda}\cdot\bar{\mu}) = (b^*, \bar{\mu})(a^*, \bar{\lambda}) =$
$= (b,\mu)^* (a,\lambda)^*$. ⌋

(b) If A is a C^*-algebra and we define
$\|(a,\lambda)\| = \max\{|\lambda|; \sup_{\substack{b \in A \\ \|b^*\| \le 1}} \|ab + \lambda b\|\}$, $(a,\lambda) \in A^+$,

then A^+ is a C^*-algebra

⌈ ① Set $p_1(a,\lambda) := |\lambda|$, $(a,\lambda) \in A^+$

Then p_1 is a seminorm
$p_1((a,\lambda)(b,\mu)) \le p_1(a,\lambda)\, p_1(b,\mu)$ [In fact "="]
$p_1((a,\lambda)^*(a,\lambda)) = p_1(a,\lambda)^2$

... this is obvious from definitions ✓

② Set $p_2(a,\lambda) = \sup_{\substack{b \in A \\ \|b\| \le 1}} \|ab + \lambda b\|$, $(a,\lambda) \in A^+$

Interpretation: define $L(a,\lambda)(b) = ab + \lambda b$, $b \in A$

Then $L(a,\lambda): A \to A$ linear, $\|L(a,\lambda)\| \le \|a\| + |\lambda|$
So, $L(a,\lambda) \in \mathcal{L}(A)$.

Moreover, $p_2(a,\lambda) = \|L(a,\lambda)\|$ by the definitions

③ $(a,\lambda) \mapsto L(a,\lambda)$ is linear (clear)

$L_{((a_1,\lambda_1)(a_2,\lambda_2))} = L_{(a_1,\lambda_1)} L_{(a_2,\lambda_2)}$

$\ulcorner$ $L_{(a_1,\lambda_1)} L_{(a_2,\lambda_2)}(b) = L_{(a_1,\lambda_1)}(a_2 b + \lambda_2 b) =$

$= a_1 a_2 b + \lambda_2 a_1 b + \lambda_1 a_2 b + \lambda_1 \lambda_2 b =$

$= L(a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1\lambda_2)(b) = L(((a_1,\lambda_1)(a_2,\lambda_2)))(b)$. $\lrcorner$

④ Thus p_2 is a seminorm and $p_2((a_1,\lambda_1)(a_2,\lambda_2)) \le$
$\le p_2(a_1,\lambda_1)\, p_2(a_2,\lambda_2)$

$\ulcorner$ This follows from ② and ③ $\lrcorner$

⑤ $p_2((a,\lambda)^*, (a,\lambda)) = p_2((a,\lambda))^2$

$\ulcorner$ It is enough to prove "$\ge$"

$p_2((a,\lambda)^*(a,\lambda)) = p_2((a^*a + \bar{\lambda}a + \lambda a^*, \bar{\lambda}\cdot\lambda)) =$

$= \sup_{\substack{b \in B \\ \|b\| \le 1}} \|a^*ab + \bar{\lambda}ab + \lambda a^* b + \bar{\lambda}\lambda b\| \ge$

$\ge \sup_{\|b\| \le 1} \|b^*a^*ab + \bar{\lambda} b^* ab + \lambda b^* a^* b + \bar{\lambda}\lambda b^* b\| \ge$

$= \sup_{\|b\| \le 1} \|(b^*a^* + \bar{\lambda} b^*)(ab + \lambda b)\| = \sup_{\|b\| \le 1} \|(ab + \lambda b)^*(ab + \lambda b)\|$

$= \sup_{\|b\| \le 1} \|ab + \lambda b\|^2 = p_2(a,\lambda)^2$ $\lrcorner$

(6) $p_2(a,0) = \|a\|$ for $a \in A$

$\ulcorner$ $p_2(a,0) = \sup_{\substack{b\in A\\ \|b\|\le 1}} \|ab\| \le \sup_{\substack{b\in A\\ \|b\|\le 1}} \|a\|\cdot\|b\| = \|a\|$

conversely: if $a=0$, the $p_2(a,0) = p_2(a\,0) = 0$

if $a \neq 0$, then

$$p_2(a,0) \ge \left\| a \cdot \frac{a^*}{\|a\|} \right\| = \|a\|,$$

so the equality follows. $\lrcorner$

(7) $\|(a,\lambda)\| = \max\{p_1(a,\lambda), p_2(a,\lambda)\}$

$\Rightarrow$ $\|\cdot\|$ is a seminorm satisfying

$\|(a_1,\lambda_1)(a_2,\lambda_2)\| \le \|(a_1,\lambda_1)\|\,\|(a_2,\lambda_2)\|$

$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2$

$\ulcorner$ By (1), (4), (5) $\lrcorner$

(8) $\|\cdot\|$ is a norm:

$\ulcorner$ $\|(a,\lambda)\|=0 \Rightarrow |\lambda| = p_1(a,\lambda) = 0$, i.e. $\lambda = 0$

$p_2(a,0) = \|(a,0)\| = \|a\|$ (by (6))

so $a = 0$. $\lrcorner$

(9) $(A^+, \|\cdot\|)$ is a C^*-algebra

$\ulcorner$ It's enough to show that $\|\cdot\|$ is complete

This follows, for example, from the fact that

$\|\cdot\|$ is equivalent to $\|\cdot\|_1$ defined by $\|(a,\lambda)\|_1 = \|a\| + |\lambda|$

$\|(a_n,\lambda_n)\|_1 \to 0 \Leftrightarrow \|a_n\| \to 0$ & $|\lambda_n| \to 0 \Leftrightarrow \|(a_n,0)\| \to 0$ & $\|(0,\lambda_n)\| \to 0$

$\Rightarrow \|(a_n,\lambda_n)\| \to 0$

Conversely: $\|(a_n,\lambda_n)\| \to 0 \Rightarrow p_1(a_n,\lambda_n) = |\lambda_n| \to 0 \Rightarrow$

$\Rightarrow \|(0,\lambda_n)\| \to 0 \Rightarrow \underbrace{\|(a_n,0)\|}_{=\|a_n\|} = \|(a_n,\lambda_n) - (0,\lambda_n)\| \to 0$ $\lrcorner$

(10) Suppose that A has no unit. Then p_2 is a norm

⌈ $p_2(a,\lambda)=0 \Rightarrow \forall b\in A \quad ab+\lambda b=0$

If $a=0$, then necessarily $\lambda=0$, so $(a,\lambda)=(0,0)$
If $a\neq 0$, then $\forall b\in A$: $b=-\frac{a}{\lambda}b \Rightarrow$

$-\frac{a}{\lambda}$ is a left unit, hence A is unital. $\{\}$ ⌋

(11) If A has no unit, then (A,p_2) is a C^*-algebra, hence $p_2=\|\cdot\|$.

⌈ The equality follows from Corollary 28. So, it is enough to show that p_2 is complete.

~~It is enough~~ Define $\theta: A^+\to\mathbb{C}$ $\quad \theta(a,\lambda)=\lambda$.
Then θ is a linear functional.
$\ker\theta=\{(a,0), a\in A\}$ ... it is closed as A is complete and p_2 is a norm
Thus θ is cts.

So, $p_2(a_n,\lambda_n)\to 0 \Rightarrow \lambda_n\to 0$.
$p_2(0,\lambda_n)=|\lambda_n|\to 0$
so $\|a_n\|=p_2(a_n,0)=p_2((a_n,\lambda_n)-(0,\lambda_n))\to 0$

Therefore p_2 is equivalent to $\|\cdot\|_1$, as in (9). ⌋

(12) If A has a unit e, then p_2 is not a norm, for example $p_2(-e,1)=0$.

Then $p_2(a,\lambda)=\|a+\lambda e\|$, hence $\|(a,\lambda)\|=\max\{|\lambda|,\|a+\lambda e\|\}$