A commutative C^*-algebra, $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ its Gelfand transform.

Then Γ is an isometric $*$-isomorphism of A onto $\mathcal{C}_0(\Delta(A))$

Proof:

① Thm 24 (g) $\Rightarrow$ $\forall x \in A$ $\|\Gamma(x)\| = r(x)$
A commutative $\Rightarrow$ each $x \in A$ is normal, thus
$r(x) = \|x\|$ by Prop. 27

It follows that Γ is an isometry of A into $\mathcal{C}_0(\Delta(A))$

② Γ is a homomorphism by Thm 24 (a)
Further, for $x \in A$, $h \in \Delta(A)$ we have
$$\Gamma(x^*)(h) = \underset{\substack{\uparrow \\ \text{Prop. 32 (c)}}}{h(x^*)} = \overline{h(x)} = \overline{\Gamma(x)(h)}$$

So, Γ is a $*$-homomorphism.

③ $\Gamma(A)$ is a subalgebra of $\mathcal{C}_0(\Delta(A))$ separating points of $\Delta(A)$ (Thm 24 (j)) and, moreover

$f \in \Gamma(A) \Rightarrow \bar{f} \in \Gamma(A)$ by ②. The Stone-Weierstrass theorem says that $\Gamma(A)$ is dense in $\mathcal{C}_0(\Delta(A))$

[more precisely: A unital $\Rightarrow$ $1 \in \Gamma(A)$, so
$\Gamma(A)$ contains constants,
so we can use S-W theorem,
for non-unital case see ④ below]

By ① we get that $\Gamma(A)$ is closed in $\mathcal{C}_0(\Delta(A))$,
so $\Gamma(A) = \mathcal{C}(\Delta(A))$ if
A is unital

④ A not unital, take $\Gamma^+ : A^+ \to \mathcal{C}(\Delta(A^+))$

we know already that Γ^+ is onto.

It follows from Thm 24 (5) that

$$\Gamma^+(\{(a,0),\ a\in A\}) = \{f \in \Gamma^+(A^+)\ ;\ f(h_\infty) = 0\}$$

So, $\Gamma(A) = \mathcal{C}_0(\Delta(A))$.

⑤ It follows that A is unital if and only if $\Delta(A)$ is compact ($\Leftrightarrow \mathcal{C}_0(\Delta(A))$ is unital)

$\boxtimes$

<u>Corollary:</u> A, B commutative C^*-algebras
A, B are $*$-isomorphic $\Leftrightarrow$ $\Delta(A)$ and $\Delta(B)$ are homeomorphic

<u>Proof:</u> $\Rightarrow$: $T: A \to B$ $*$-isomorphism onto
By Prop. 30 $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$,
so T is an isometry
Thus $T' : B^* \to A^*$ is a w^*-w^* homeomorphism
Moreover, $T'(\Delta(B)) = \Delta(A)$ as
T is an isomorphism of Banach algebras

$\Leftarrow$ $\Delta(A)$ homeomorphic to $\Delta(B)$ $\Rightarrow$ $\mathcal{C}_0(\Delta(A))$ is $*$-isomorphic to $\mathcal{C}_0(\Delta(B))$

Hence $A \simeq \mathcal{C}_0(\Delta(A))$ is $*$-isomorphic

to $B \simeq \mathcal{C}_0(\Delta(B))$