Corollary: A, B C^*-algebras, $h: A \to B$ a one-to-one $*$-homomorphism $\Rightarrow$ h is an isometry of A into B

Proof: (1) WLOG A, B unital and $h(e) = e$

$\ulcorner$ otherwise take $h^+: A^+ \to B^+$ defined by $h^+(a,\lambda) = (h(a), \lambda)$ $\lrcorner$

(2) It is enough to show $\|h(x)\| = \|x\|$ for x self-adjoint

$\ulcorner$ $y \in A$ general $\Rightarrow \|h(y)\|^2 = \|h(y)^* h(y)\| =$

$$= \|h(y^* y)\| \underset{\substack{\uparrow \\ y^*y \text{ self-adjoint}}}{=} \|y^* y\| = \|y\|^2 \lrcorner$$

(3) Let $x \in A$ be self-adjoint. Then $h(x) \in B$ is also self-adjoint

Set $A_0 := \overline{\mathrm{alg}}\{e, x\} = \overline{\mathrm{span}}\{e, x, x^2, x^3, \dots\}$
$B_0 := \overline{\mathrm{alg}}\{e, h(x)\} = \overline{\mathrm{span}}\{e, h(x), h(x)^2, h(x)^3, \dots\}$

Then A_0, B_0 are unital commutative C^*-subalgebras of A, B and, moreover, $h(A_0) \subset B_0$

By Gelfand-Naimark $A_0 \simeq \mathcal{C}(\Delta(A_0))$, $B_0 \simeq \mathcal{C}(\Delta(B_0))$, hence $h\restriction A_0$ defines a one-to-one $*$-homomorphism

$$\tilde{h} : \mathcal{C}(\Delta(A_0)) \to \mathcal{C}(\Delta(B_0))$$

s.t. $\tilde{h}(1) = 1$

By Example 31 $\tilde{h}$ is an isometry.

In particular $\|h(x)\| = \|x\|$.