FUGLEDE'S THEOREM

Let A be a C^*-algebra, $x \in A$ normal.
Let $y \in A$ commute with x. Then y commutes with x^*

WLOG A unital

Proof: Preparation: Let $a \in A$

Then $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ (by Thm 17 (c, e))

Hence $\exp(a^*) = (\exp(a))^*$

Further, if $ab = ba$, then $\exp(a+b) = \exp(a)\exp(b)$

[$ab = ba \Rightarrow (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$]

In particular, $\exp(a)\cdot\exp(-a) = \exp(0) = e$
So, $\exp(a)^{-1} = \exp(-a)$.

Further, $ya = ay \Rightarrow y\exp(a) = \exp(a)y$ (Thm 17 (c))

The proof itself

(1) $yx = xy \Rightarrow \forall \lambda \in \mathbb{C}\ \ y e^{\lambda x} = e^{\lambda x} y$, so $y = e^{-\lambda x} y e^{\lambda x}$

(2) Set $f(\lambda) := e^{\lambda x^*} y e^{-\lambda x^*}$, $\lambda \in \mathbb{C}$
Then $\forall \varphi \in A^*$: $\varphi \circ f$ is an entire function, as it can be expressed using a power series in $\mathbb{C}$

(3) by (1) for $\lambda \in \mathbb{C}$ we have $f(\lambda) = e^{\lambda x^*} y e^{-\lambda x^*} =$
$= e^{\lambda x^*} e^{-\bar\lambda x} y e^{\bar\lambda x} e^{-\lambda x^*} = e^{\lambda x^* - \bar\lambda x} y e^{\bar\lambda x - \lambda x^*}$

λx^*, $\bar\lambda x$ commute, as x is normal

(4) $\left(e^{\lambda x^* - \bar\lambda x}\right)^* = e^{\bar\lambda x - \lambda x^*} = \left(e^{\lambda x^* - \bar\lambda x}\right)^{-1}$

$\Rightarrow \|e^{\lambda x^* - \bar\lambda x}\| = 1$

(5) By (3) and (4) we get $\|f(\lambda)\| \leq \|y\|$, $\lambda \in \mathbb{C}$

So, for each $\varphi \in A^*$ $\varphi \circ f$ is a bdd entire function, so $\varphi \circ f$ is constant by the Liouville theorem.

It follows that f is constant (using H-B thm)

Thus $\forall \lambda \in \mathbb{C}$: $f(\lambda) = f(0) = y$

So, $\forall \lambda \in \mathbb{C}$: $y = e^{\lambda t^*} y e^{-\lambda t^*}$, thus

$$y e^{\lambda t^*} = e^{\lambda t^*} y, \quad \lambda \in \mathbb{C}$$

$$y \cdot \sum_{n=0}^{\infty} \frac{\lambda^n (t^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^n (t^*)^n}{n!} \cdot y, \quad \lambda \in \mathbb{C}$$

~~For $\lambda \neq 0$~~ For $n=0$ we have y on both sides, so

$$y \cdot \sum_{n=1}^{\infty} \frac{\lambda^n (t^*)^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^n (t^*)^n}{n!} y, \quad \lambda \in \mathbb{C}$$

Divide by λ, then

$$y \sum_{n=1}^{\infty} \frac{\lambda^{n-1} (t^*)^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} (t^*)^n}{n!} y, \quad \lambda \in \mathbb{C} \setminus \{0\}$$

by continuity also for $\lambda = 0$. Insert $\lambda = 0$ and we get

$$y t^* = t^* y.$$