CONTINUOUS FUNCTIONAL CALCULUS

Let A be a unital C^*-algebra, e be its unit and $x \in A$ be a normal element. (i.e., $x^*x = xx^*$)

(1) Let $B := \overline{alg}\{e, x, x^*\}$

Then clearly B is a commutative C^*-algebra,
B is a C^*-subalgebra of A
$e \in B$, it is also unit of B

Thus, by Prop. 36 $\forall y \in B : \sigma_B(y) = \sigma_A(y)$. So, we will write only $\sigma(y)$.

(2) Let $h = \hat{x}$; i.e. $h(\varphi) = \varphi(x)$, $\varphi \in \Delta(B)$
Then h is a homeomorphism of $\Delta(B)$ onto $\sigma(x)$

Γ • $h = \hat{x}$ is cts on $\Delta(B)$
• $h(\Delta(B)) = \sigma(x)$ by Thm 24 (e)
• h is one-to-one:
$\varphi_1, \varphi_2 \in \Delta(B)$: $h(\varphi_1) = h(\varphi_2)$
Then: $\varphi_1(e) = \varphi_2(e) = 1$
$\varphi_1(x) = \varphi_2(x)$
$\varphi_1(x^*) = \overline{\varphi_1(x)} = \overline{\varphi_2(x)} = \varphi_2(x^*)$

Prop. 32 (c)

$\{y \in B \,;\, \varphi_1(y) = \varphi_2(y)\}$ is a closed algebra containing e, x, x^*, so it equals B

Hence $\varphi_1 = \varphi_2$

• $\Delta(B)$ compact, h cts and one-to-one $\Rightarrow$ h is a homeomorphism $\lrcorner$

③ Let $\Gamma : B \to \mathcal{C}(\Delta(B))$ be the Gelfand transform of B

For $f \in \mathcal{C}(\sigma(x))$ define $\tilde{f}(x) := \Gamma^{-1}(f \circ h)$

④ $\Phi : f \mapsto \tilde{f}(x)$ is an isometric $*$-isomorphism of $\mathcal{C}(\sigma(x))$ onto B

⌈ h is a homeomorphism $\Rightarrow$ $f \mapsto f\circ h$ is an isometric $*$-isomorphism of $\mathcal{C}(\sigma(x))$ onto $\mathcal{C}(\Delta(B))$

Γ is an isometric $*$-isomorphism of B onto $\mathcal{C}(\Delta(B))$ by Theorem 33

So, Φ is such, as a composition of two such maps. ⌋

⑤ $\tilde{1}(x) = e$ (Φ preserves the unit)

$\widetilde{id}(x) = x$ ($\Gamma(x) = \hat{x} = h = id \circ h$)

⑥ p is a polynomial $\Rightarrow$ $\tilde{p}(x) = p(x)$

⌈ This follows from ④ and ⑤ ⌋

⑦ $\sigma(\tilde{f}(x)) = f(\sigma(x))$ for $f \in \mathcal{C}(\sigma(x))$.

⌈ Φ is an isometric $*$-isomorphism $\Rightarrow$ Φ preserves spectrum

$\Rightarrow$ $\sigma(\tilde{f}(x)) = \sigma(\Phi(f)) = \sigma(f) = f(\sigma(x))$ ⌋

⑧ If $y \in A$ commutes with x, it commutes with $\tilde{f}(x)$ for each $f \in \mathcal{C}(\sigma(x))$

⌈ $\{z \in A ; zy = yz\}$ is a closed subalgebra of A containing e, x and also x^* (by Thm 37)

So, it contains B. ⌋