Proof of Prop. IX.2: X ... a Banach space, $T \in \mathcal{L}(X)$

(a) $\sigma_p(T) \subset \sigma_{ap}(T)$

$\ulcorner$ OBVIOUS: $\lambda \in \sigma_p(T) \Rightarrow \exists x \in X \quad \|x\|=1 \quad Tx = \lambda x$

Take $x_n = x$ for $n \in \mathbb{N}$. Then $(Tx_n - \lambda x_n) = 0 \to 0$,

so $\lambda \in \sigma_{ap}(T)$ $\lrcorner$

(b) $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \Leftrightarrow \lambda I - T$ is an isomorphism of X into X

$\ulcorner \lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \Leftrightarrow \exists c > 0 \ \forall x \in S_X : \|(\lambda I - T)x\| \geq c$

$\Leftrightarrow \exists c > 0 \ \forall x \in X : \|(\lambda I - T)x\| \geq c\|x\|$

$\Leftrightarrow \lambda I - T$ is an into isomorphism $\lrcorner$

(c) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$

$\ulcorner \supset$ obvious.

$\subset$: $\lambda \notin \sigma_{ap}(T) \cup \sigma_r(T) \overset{(b)}{\Rightarrow} \lambda I - T$ is an isomorphism of X into X,

hence its range is closed.

In part. $\lambda I - T$ is one-to-one, hence $R(\lambda I - T)$ is dense

(as $\lambda \notin \sigma_r(T)$).

Thus $R(\lambda I - T) = X$ (being closed and dense),

thus $\lambda I - T$ is invertible. So $\lambda \notin \sigma(T)$ $\lrcorner$

(d) $\sigma_c(T) = \sigma_{ap}(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$

$\ulcorner$ ① $\lambda \in \sigma_c(T) \Rightarrow$
- $\lambda I - T$ is one-to-one, hence $\lambda \notin \sigma_p(T)$
- $R(\lambda I - T)$ is dense, hence $\lambda \notin \sigma_r(T)$
- $\lambda \in \sigma(T) \setminus \sigma_r(T) \Rightarrow \lambda \in \sigma_{ap}(T)$ by (c)

② $\lambda \in \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) \Rightarrow$
- $\lambda \notin \sigma_p(T) \Rightarrow \lambda I - T$ is one-to-one

  As $\lambda \in \sigma(T)$, $\lambda I - T$ is not onto
- $\lambda \notin \sigma_r(T)$, $\lambda I - T$ one-to-one $\Rightarrow$

  $R(\lambda I - T)$ is dense.

  Hence $\lambda \in \sigma_c(T)$. $\lrcorner$

(e) $\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T) \iff \lambda I - T$ is an isomorphism of X onto a proper closed subspace of X

[$\Rightarrow$ $\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T) \Rightarrow$ • $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \overset{(b)}{\Rightarrow} \lambda I - T$ is an isomorphism of X into X. In part. $R(\lambda I - T)$ is closed and $\lambda I - T$ is one-to-one

• $\lambda \in \sigma_r(T)$, ~~$\lambda I - T$ is one-to-one~~ $\Rightarrow R(\lambda I - T)$ is not dense

Hence $R(\lambda I - T)$ is a proper closed subspace of X

$\Leftarrow$ $\lambda I - T$ is an isomorphism of X into $X \overset{(b)}{\Rightarrow} \lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$

Moreover, $\lambda I - T$ is one-to-one and $R(\lambda I - T)$ is not dense, so $\lambda \in \sigma_r(T)$]