Lemma IX.3 (Polarization formula for operators)

H -- a Hilbert space, $T \in \mathcal{L}(H)$, $x, y \in H$:

$$\langle T(x+y), x+y\rangle = \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle$$
$$\langle T(x-y), x-y\rangle = \langle Tx, x\rangle - \langle Tx, y\rangle - \langle Ty, x\rangle + \langle Ty, y\rangle$$

Substrack:

$$(*) \quad \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 2\left(\langle Tx, y\rangle + \langle Ty, x\rangle\right)$$

Apply this formula to x and iy:

$$\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle = 2\left(\langle Tx, iy\rangle + \langle T(iy), x\rangle\right) =$$
$$= 2i\left(-\langle Tx, y\rangle + \langle Ty, x\rangle\right)$$

So, $i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle = 2\left(\langle Tx, y\rangle - \langle Ty, x\rangle\right) \quad (**)$

Add $(*)$ and $(**)$:

$$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle$$
$$\|$$
$$4\langle Tx, y\rangle.$$

Proposition IX.4 (properties of the numerical radius)

H ... a Hilbert space

For $T \in \mathcal{L}(H)$ set

$$W(T) = \{ \langle Tx, x \rangle \; ; \; x \in S_H \} \qquad \text{(numerical range)}$$
$$w(T) = \sup \{ |\lambda| \; ; \; \lambda \in W(T) \} \qquad \text{... (numerical radius)}$$

(a) w is an equivalent norm, $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ for $T \in \mathcal{L}(H)$

⌈ Clearly • $w(T) \geq 0$, $w(0) = 0$

• $x \in S_H \Rightarrow |\langle Tx, x\rangle| \leq \|Tx\| \cdot \|x\| \leq \|T\|$,
so $w(T) \leq \|T\|$, in part. $w(T) \in [0, \infty)$

• $w(\alpha T) = |\alpha| \, w(T)$ $\qquad T \in \mathcal{L}(H), \alpha \in \mathbb{C}$
$w(S+T) \leq w(S) + w(T)$ $\qquad S, T \in \mathcal{L}(H)$

So, w is a seminorm. It remains to prove $\frac{1}{2}\|T\| \leq w(T)$

For $x, y \in S_H$ we have, due to Lemma 3

$$|\langle Tx, y\rangle| \leq \frac{1}{4} \big(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| + |\langle T(x+iy), x+iy\rangle| + |\langle T(x-iy), x-iy\rangle| \big) \leq (*)$$

⌈ $|\langle T(x+y), x+y\rangle| \leq w(T) \cdot \|x+y\|^2$

• if $x+y=0$, obvious

• $x+y \neq 0$ ... $|\langle T(x+y), x+y\rangle| = \underbrace{\left|\left\langle T\left(\frac{x+y}{\|x+y\|}\right), \frac{x+y}{\|x+y\|}\right\rangle\right|}_{\leq w(T)} \|x+y\|^2$ ⌋

$$(*) \leq \frac{1}{4} \left(w(T) \left(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2 \right) \right) =$$

$$= \frac{1}{4} w(T) \left(2(\|x\|^2 + \|y\|^2) + 2(\|x\|^2 + \|iy\|^2) \right) = 2 w(T)$$

the paralelogram rule

Thus $|\langle Tx, y\rangle| \leq 2w(T)$ $\quad \forall y \in S_H$

It follows $\|T\| \leq 2w(T)$ $\lrcorner$

(b) $\langle Tx, x\rangle = 0$ for $x \in H$ $\Rightarrow$ $T = 0$

$\ulcorner$ $\forall x \in H: \langle Tx, x\rangle = 0 \Rightarrow w(T) = 0 \overset{(a)}{\Longrightarrow} \|T\| = 0 \Rightarrow T = 0$ $\lrcorner$

(c) $\langle Tx, x\rangle = \langle Sx, x\rangle$ for $x \in H \Rightarrow T = S$

$\ulcorner$ Apply (b) to $T - S$ $\lrcorner$

(d) $W(T)$ is a connected subset of $\mathbb{C}$ for $T \in \mathcal{L}(H)$

$\ulcorner$ S_H is connected and $x \mapsto \langle Tx, x\rangle$ is cts. $\lrcorner$

(e) $\sigma_p(T) \subset W(T)$, $\sigma(T) \subset \overline{W(T)}$

$\ulcorner$ $\lambda \in \sigma_p(T) \Rightarrow \exists x \in S_H$ $\lambda x = Tx \Rightarrow \langle Tx, x\rangle = \langle \lambda x, x\rangle = \lambda$
$\Rightarrow \lambda \in W(T)$

$\lambda \in \sigma(T) \overset{\text{Prop 2 (c)}}{\Longrightarrow} \lambda \in \sigma_{ap}(T)$ or $\lambda \in \sigma_r(T)$

$\lambda \in \sigma_{ap}(T) \Rightarrow \exists (x_n) \subset S_H$ $\|\lambda x_n - Tx_n\| \to 0$

So $\underbrace{\langle \lambda x_n - Tx_n, x_n\rangle}_{= \lambda - \langle Tx_n, x_n\rangle} \to 0$ $\Big\}$ $\Rightarrow \langle Tx_n, x_n\rangle \to \lambda$
$\Rightarrow \lambda \in \overline{W(T)}$

$\lambda \in \sigma_r(T) \Rightarrow R(\lambda I - T)$ is not dense $\Rightarrow \exists x \in S_H$ $x \perp R(\lambda I - T)$
So, in particular $\langle \lambda x - Tx, x\rangle = 0$, so $\langle Tx, x\rangle = \lambda$
$\Rightarrow \lambda \in W(T)$ $\lrcorner$

(f) $w(T) \geq r(T)$
$\ulcorner$ follows from (e) $\lrcorner$