Theorem 9 (Hilbert-Schmidt theorem)

Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a compact normal operator.

If $T=0$, then $\ker T = H$, so any ON basis is formed by eigenvectors

Suppose $T \neq 0$. As T is compact, we know from the Introduction to FA that

- $\lambda \in \sigma(T) \setminus \{0\} \Rightarrow \lambda$ is an eigenvalue and $\ker(\lambda I - T)$ has finite dimension

- $\sigma(T) \setminus \{0\}$ is either finite or it is a sequence converging to 0.

So, let $\sigma(T) \setminus \{0\} = \{\mu_j, j \in J\}$, where $J = \mathbb{N}$ or $J = \{1, \dots, l\}$ for some $l \in \mathbb{N}$

and $\mu_j, j \in J$ are distinct.

Set $H_j := \ker(\mu_j I - T)$, $j \in J$

By Prop. 5(d) we know $H_j \perp H_k$ for $j \neq k$

Moreover, for $x \in H_j$ we have $Tx = \mu_j x \in H_j$

and $T^* x = \overline{\mu_j} x \in H_j$. (Prop. 5(c))

So, $T(H_j) \subset H_j$, $T^*(H_j) \subset H_j$.

Set $H_0 := \left(\bigcup_{j \in J} H_j \right)^\perp$. Then $T(H_0) \subset H_0$ and $T^*(H_0) \subset H_0$

Indeed, let $x \in H_0$. Suppose $y \in H_j$ for some $j \in J$. Then

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = 0 \quad \text{as } T^*y \in H_j$$
$$\langle T^*x, y\rangle = \langle x, Ty\rangle = 0 \quad \text{as } Ty \in H_j.$$

Therefore, indeed, ~~Tx~~ $Tx, T^*x \in H_0$.

Consider the operator $T\restriction_{H_0} \in L(H_0)$. Then $(T\restriction_{H_0})^* = T^*\restriction_{H_0}$,

so $T\restriction_{H_0}$ is normal. Moreover, $T\restriction_{H_0}$ is compact and

$\sigma_p(T\restriction_{H_0}) \setminus \{0\} = \emptyset$. It follows $\sigma(T\restriction_{H_0}) \setminus \{0\} = \emptyset$,
hence $r(T\restriction_{H_0}) = 0$, so $\|T\restriction_{H_0}\| = 0$, thus $T\restriction_{H_0} = 0$.

In other words, $H_0 = \operatorname{Ker} T$. By Prop. 5(d) we have $H_0 \perp H_j$ for $j \in J$. If we put together ON bases of H_j, $j \in J \cup \{0\}$, we obtain an ON basis of H made of eigenvectors of T.

For $j \in J$ fix an ON basis of H_j: $y^j_1, \dots, y^j_{n_j}$

Set $N = \mathbb{N}$ if $J = \mathbb{N}$ and $N = \{1, \dots, n_1 + \dots + n_e\}$ otherwise

Let $(x_j)_{j \in N} = y^1_1, \dots, y^1_{n_1}, y^2_1, \dots, y^2_{n_2}, \dots$

$(\lambda_j)_{j \in N} = \underbrace{\mu_1, \dots, \mu_1}_{n_1 \text{ times}}, \underbrace{\mu_2, \dots, \mu_2}_{n_2 \text{ times}}, \dots$

Then for $x \in H$ we have $x = x_0 + \sum_{j \in N} \langle x, x_j\rangle x_j$, where $x_0 \in H_0$

Then $Tx = \underbrace{Tx_0}_{0} + \sum_{j \in N} \langle x, x_j\rangle \underbrace{Tx_j}_{\lambda_j x_j} = \sum_{j \in N} \lambda_j \langle x, x_j\rangle x_j$.

Prop. 10 H .. a Hilbert space of infinite dimension

$T \in L(H)$ compact normal operator

$Tx = \sum_{k \in N} \lambda_k \langle x, x_k \rangle x_k$, $x \in H$, where $(x_k)_{k \in N}$ is an ON system.

Then $\sigma(T) = \{0\} \cup \{\lambda_k, k \in \mathbb{N}\}$

($\subset$: by compactness of T $\supset$: λ_k are eigenvalues
0 by $\dim H = \infty$)

$f \in C(\sigma(T))$

Then $\tilde{f}(T)x = f(0)x + \sum_{k \in N} (f(\lambda_k) - f(0)) \langle x, x_k \rangle x_k$

$= f(0) Px + \sum_{k \in N} f(\lambda_k) \langle x, x_k \rangle x_k$

where P is the OG projection on $\operatorname{Ker} T$

Denote $f^{\square}(T)x = f(0) Px + \sum_{k \in N} f(\lambda_k) \langle x, x_k \rangle x_k$

$f^{\square}(T)x$ is well-defined by Bessel-inequality
as f is bdd.

Moreover $\|f^{\square}(T)x\| \leq \|f\|_\infty \cdot \|x\|$
(Bessel inequality)

Clearly $f^{\square}(T)$ is a linear operator, hence $f^{\square}(T) \in L(H)$

Clearly $1^{\square}(T) = I$, $id^{\square}(T) = T$

Further, $(f^{\square}(T))^* = \bar{f}^{\square}(T)$

Indeed, let $x, y \in H$

$$\langle x, f^{\square}(T) y\rangle = \langle Px + \sum_{k\in N} \langle x, x_k\rangle x_k, f(0)Py + \sum_{k\in N} f(\lambda_k)\langle y, x_k\rangle x_k\rangle$$

$$= \langle Px, f(0)Py\rangle + \sum_{k\in N} \langle x, x_k\rangle \overline{f(\lambda_k)}\, \overline{\langle y, x_k\rangle} =$$

$$= \langle \overline{f(0)}Px, Py\rangle + \sum_{k\in N} \overline{f(\lambda_k)}\langle x, x_k\rangle \overline{\langle y, x_k\rangle} =$$

$$= \langle \overline{f(0)}P(x) + \sum_{k\in N} \overline{f(\lambda_k)}\langle x, x_k\rangle x_k, Py + \sum_{k\in N}\langle y, x_k\rangle x_k\rangle = \langle \bar{f}^{\square}(T)x, y\rangle$$

$f \mapsto f^{\square}(T)$ is linear [clear]

$f \mapsto f^{\square}(T)$ is multiplicative

$$f^{\square}(T)\, g^{\square}(T) x = f(0)\, P(g^{\square}(T)x) + \sum_{k\in N} f(\lambda_k)\langle g^{\square}(T)x, x_k\rangle x_k$$

$$= f(0)g(0)\, Px + \sum_{k\in N} f(\lambda_k)\, g(\lambda_k)\, \langle x, x_k\rangle x_k = (fg)^{\square}(T)(x)$$

It follows that $f^{\square}(T) = \tilde{f}(T)$, $f \in \mathcal{C}(\sigma(T))$

$\tilde{f}(T)$ compact $\iff f(0) = 0$

$\Leftarrow$ $\tilde{f}(T)x = \sum_{k\in N} \underbrace{f(\lambda_k)}_{\to 0} \langle x, x_k\rangle x_k$

$\Rightarrow$ $\tilde{f}(T)$ can so approximated by finite rank operators

$\Rightarrow$ $f(0) \neq 0 \Rightarrow \tilde{f}(T) - f(0)I$ is compact $\Rightarrow$ $\tilde{f}(T)$ is not compact.

$$\tilde{f}(T) - f(0)I = \widetilde{(f - f(0))}(T)$$

Theorem 11 $T \in L(H)$, T nonzero compact operator

$\Rightarrow$ T^*T is also compact and nonzero
($\|T^*T\| = \|T\|^2 \neq 0$)

So $|T| = \sqrt{T^*T}$ is also compact (by Prop. 10 as $\sqrt{0} = 0$)

$$|T|x = \sum_{k \in \mathbb{N}} \alpha_k \langle x, e_k \rangle e_k, \quad x \in H \quad \text{by Thm 9}$$

$\alpha_k > 0$ as $\alpha_k \in \sigma(|T|) \setminus \{0\}$, $\sigma(|T|) \subset [0, \infty)$

$T = U|T|$ polar decomposition.

Then $$Tx = \sum_{k \in \mathbb{N}} \alpha_k \langle x, e_k \rangle \underbrace{Ue_k}_{f_k}$$

U partial isometry $\Rightarrow$ $(Ue_k)_{k \in \mathbb{N}}$ orthonormal system