Let (f_n) be a sequence of strongly measurable functions $f_n : M \to X$ s.t. $f_n \to f$ pointwise.
Then f is strongly measurable

Proof: Let $u_{m,n}$ be simple measurable such that $u_{m,n} \xrightarrow{m} f_n$ pointwise for each $n \in \mathbb{N}$.

Let $C := \bigcup_{m,n} u_{m,n}(M) \Rightarrow C$ is a countable set

Enumerate it $C = \{x_k ; k \in \mathbb{N}\}$

For $k \in \mathbb{N}$ define $g_k : M \to X$ by

$g_k(t) =$ the point from $\{x_1, \dots, x_k\}$ with the smallest distance to $f(t)$. If there are more points with the same distance, take the one with smallest index

i.e. $g_k(t) = x_j \iff \forall i \in \{1, \dots k\} : \|x_j - f(t)\| \le \|x_i - f(t)\|$
$\quad \& \ \forall i \in \{1, \dots, k\}, i < j : \|x_j - f(t)\| < \|x_i - f(t)\|$

Then g_k is a simple function $\quad (g(M) \subset \{x_1, \dots, x_k\})$

$g_k \to f$ pointwise

$\Gamma t \in M, \ \varepsilon > 0 \quad \Rightarrow \quad \exists n_0 \ \forall n \ge n_0 \ \|f_n(t) - f(t)\| < \frac{\varepsilon}{2}$

Fix one $n \ge n_0$ Then there is m_0 s.t.
$\forall m \ge m_0 \ \|u_{m,n}(t) - f_n(t)\| < \frac{\varepsilon}{2}$

Fix one $m \ge m_0$. Then $\|u_{m,n}(t) - f(t)\| < \varepsilon$

There is k_0 s.t. $u_{m,n}(t) = x_{k_0}$.
Then for $k \ge k_0$ $\|g_k(t) - f(t)\| < \varepsilon$ $\lrcorner$

Moreover, g_k are measurable. To show it, it is enough to show that

$$g_k^{-1}(x_j) \in \mathcal{A} \text{ for } j = 1 \dots k$$

<u>Auxilliary observation:</u>

f is Borel $\mathcal{A}$-measurable

(by Prop. 1 (c, b))

$\Rightarrow \forall x \in X$ $f - x$ is Borel $\mathcal{A}$-measurable (easy)

$\Rightarrow \forall x \in X$: $t \mapsto \|f(t) - x\|$ is $\mathcal{A}$-measurable

(Prop. 1 (e))

<u>Now</u>: $g(t) = x_j \iff \forall i \in \{1 \dots k\}$ $\|x_j - f(t)\| \leq \|x_i - f(t)\|$
$\&$ $\forall i \in \{1, \dots k\}, i < j : \|x_j - f(t)\| < \|x_i - f(t)\|$

$\iff \forall i \in \{1, \dots, k\}$ $\forall q \in \mathbb{Q}$:
$\|x_j - f(t)\| \leq q$ or $\|x_i - f(t)\| \geq q$

$\&$

$\forall i \in \{1, \dots k\}, i < j$ $\exists q \in \mathbb{Q}$
$\|x_j - f(t)\| < q < \|x_i - f(t)\|$

Hence

$$g_k^{-1}(x_j) = \bigcap_{i=1}^{k} \bigcap_{q \in \mathbb{Q}} \left(\{t ; \|x_j - f(t)\| \leq q\} \cup \{t ; \|x_i - f(t)\| \geq q\} \right)$$

$$\cap \bigcap_{i=1}^{j-1} \bigcup_{q \in \mathbb{Q}} \left(\{t ; \|x_j - f(t)\| < q\} \cap \{t ; \|x_i - f(t)\| > q\} \right)$$

and this set belongs to $\mathcal{A}$.