

VIII.5 C*-algebras – basic properties

Definition. Let A be a Banach algebra.

- An **involution** on A is a mapping $x \mapsto x^*$ of A into itself such that for each $x, y \in A$ and $\lambda \in \mathbb{C}$ one has

$$(x + y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda}x^*, \quad (xy)^* = y^*x^* \quad \text{and} \quad x^{**} = x.$$

- A Banach algebra A with involution is called a **C^* -algebra** if for each $x \in A$ one has

$$\|x^*x\| = \|x\|^2.$$

- If A is a Banach algebra with involution and $x \in A$, the element x is called **selfadjoint** (or **hermitien**) if $x^* = x$; x is called **normal** if $x^*x = xx^*$.

Remarks.

- (1) Let A be a Banach algebra with involution. Then $e \in A$ is a left unit if and only if e^* is a right unit. Hence, if A has either a left unit or a right unit, it is unital and the unit is selfadjoint.
- (2) If A is a Banach algebra with involution such that

$$\|x^*x\| \geq \|x\|^2 \quad \text{for } x \in A,$$

then A is a C^* -algebra.

- (3) Let A be a C^* -algebra. Then $x \mapsto x^*$ is a conjugate linear isometry of A onto A . Hence,

$$\|x^*x\| = \|xx^*\| = \|x\|^2 = \|x^*\|^2 \quad \text{for } x \in A.$$

Examples 25.

- (1) The complex field is a commutative C^* -algebra, if the involution is defined by $\lambda^* = \bar{\lambda}$ for $\lambda \in \mathbb{C}$.
- (2) The algebra $C_0(T)$ (where T is locally compact space) is a commutative C^* -algebra, if the involution is defined by $f^*(t) = \overline{f(t)}$ for $t \in T$.
- (3) The matrix algebra M_n is a C^* -algebra if the involution is defined by

$$\left((a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \right)^* = (\bar{a}_{ji})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}.$$

- (4) If H is a Hilbert space, then the algebras $L(H)$ and $K(H)$ are C^* -algebras, if the involution T^* is defined to be the adjoint operator to T .
- (5) On the algebra $L^1(\mathbb{R}^n)$ one can define an involution by $f^*(x) = \overline{f(x)}$, $x \in \mathbb{R}^n$; or by $f^*(x) = \overline{f(-x)}$, $x \in \mathbb{R}^n$. $L^1(\mathbb{R}^n)$ is not a C^* -algebra with any of these involutions.

Proposition 26 (properties of algebras with involution). Let A be a Banach algebra with involution and let $x \in A$. Then:

- (a) Elements $x + x^*$, $i(x - x^*)$, x^*x are selfadjoint.
- (b) There exist uniquely determined selfadjoint elements $u, v \in A$ such that $x = u + iv$. Moreover, x is normal if and only if $uv = vu$.
- (c) If A is unital, then $x \in G(A)$ if and only if $x^* \in G(A)$ (then $(x^*)^{-1} = (x^{-1})^*$).
- (d) $\sigma(x^*) = \{\bar{\lambda} : \lambda \in \sigma(x)\}$.

Proposition 27 (on the spectral radius and the norm of a normal element). *If A is a C^* -algebra and $a \in A$ is normal, then $r(a) = \|a\|$.*

Corollary 28. *Let A be an algebra with involution. Then there is at most one norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a C^* -algebra.*

Proposition 29 (adding a unit). *Let A be a Banach algebra with involution.*

- (a) A^+ is again a Banach algebra with involution, provided the involution is defined by $(a, \lambda)^* = (a^*, \bar{\lambda})$ for $(a, \lambda) \in A^+$.
- (b) If A is a C^* -algebra, then A^+ is also a C^* -algebra, if the involution is defined as in (a) and the norm on A^+ is defined by

$$\|(a, \lambda)\| = \max\{|\lambda|, \sup\{\|ab + \lambda b\|; b \in A, \|b\| \leq 1\}\}.$$

- (c) If A is a C^* -algebra with no unit, then the norm defined in (b) can be expressed as

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\|; b \in A, \|b\| \leq 1\}.$$

Remark: The norm on A^+ defined in Proposition 29(b) differs from the norm given in Proposition 2(b). It follows from Corollary 28 that the formula from Proposition 29(b) is the unique possible.

Definition. Let A and B be C^* -algebras and let $h : B \rightarrow A$. We say that h is a **$*$ -homomorphism**, if it is a homomorphism of Banach algebras satisfying moreover $h(x^*) = h(x)^*$ for each $x \in B$.

Proposition 30 (on the automatic continuity of a $*$ -homomorphism). *Let A and B be C^* -algebras and let $h : B \rightarrow A$ be a $*$ -homomorphism of B into A . Then $\|h\| \leq 1$.*

Example 31. *Let K, L be compact Hausdorff spaces and let $\varphi : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$ be a $*$ -homomorphism satisfying $\varphi(1) = 1$. Then there is a continuous mapping $\alpha : L \rightarrow K$ such that $\varphi(f) = f \circ \alpha$ for $f \in \mathcal{C}(K)$. If φ is moreover one-to-one, then $\alpha(L) = K$, so φ is an isometry of $\mathcal{C}(K)$ into $\mathcal{C}(L)$.*

Proposition 32. *Let A be a C^* -algebra and let $a \in A$. Then:*

- (a) *If a is selfadjoint, then $\sigma(a) \subset \mathbb{R}$.*
- (b) *If A is unital and $a^* = a^{-1}$ (i.e., a is **unitary**), then $\sigma(a) \subset \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*
- (c) *For $h \in \Delta(A)$ and $a \in A$ one has $h(a^*) = \overline{h(a)}$ (i.e., h is a $*$ -homomorphism whenever it is a homomorphism).*

Theorem 33 (Gelfand-Naimark). *Let A be a commutative C^* -algebra and let $\Gamma : A \rightarrow C_0(\Delta(A))$ be its Gelfand transform. Then Γ is an isometric $*$ -isomorphism of the C^* -algebra A onto the C^* -algebra $C_0(\Delta(A))$ (in particular $\widehat{x^*} = \overline{\widehat{x}}$ for $x \in A$).*

Therefore, A is unital if and only if $\Delta(A)$ is compact.

Corollary 34. *Let A and B be commutative C^* -algebras. Then A and B are $*$ -isomorphic if and only if $\Delta(A)$ and $\Delta(B)$ are homeomorphic.*

Corollary 35. *Let A and B be C^* -algebras and let $h : B \rightarrow A$ be a one-to-one $*$ -homomorphism of B into A . Then h is an isometry of B into A .*