IX. Operators on a Hilbert space

Convention. In this chapter we consider the Banach spaces over the complex field (unless the converse is explicitly stated). In particular, the Hilbert spaces we deal with are the complex ones.

IX.1 Various types of operators and their properties

Reminder: Let $H$ and $K$ be Hilbert spaces.

1. By $L(H, K)$ we denote the Banach space of all the bounded linear operators $T : H \to K$ equipped with the operator norm. $L(H)$ is a shortcut for $L(H, H)$.

2. For any $T \in L(H, K)$ there is a unique operator $T^* \in L(K, H)$, called the adjoint of $T$ satisfying
   $$\langle Tx, y \rangle_K = \langle x, T^*y \rangle_H$$ for $x \in H$ and $y \in K$.

3. The mapping $T \mapsto T^*$ is an involution on $L(H)$ it turns $L(H)$ to be a $C^*$-algebra. Thus the notions and the results from Chapter VIII could be applied to $L(H)$. This applies, in particular, to the notions of spectrum, spectral radius, resolvent set, resolvent function, holomorphic functional calculus, self-adjoint, normal and unitary elements and continuous functional calculus for normal elements.

4. For $x, y \in H$ the following polarization identity holds:
   $$\langle x, y \rangle = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2 \right).$$

Definition. Let $H$ and $K$ be Hilbert spaces. An operator $T \in L(H, K)$ is called unitary if $T^* = T^{-1}$, i.e., if $T^*T = I_H$ and $TT^* = I_K$.

Proposition 1 (a characterization of unitary operators). Let $H$ and $K$ be Hilbert spaces and $T \in L(H, K)$. Consider the following assertions:
   
   (i) $T$ is unitary.
   (ii) $T$ is an isometry of $H$ onto $K$.
   (iii) $T$ is an isometry of $H$ into $K$.
   (iv) $\{Tx, Ty\}_K = \{x, y\}_H$ for $x, y \in H$.

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). If $T$ is assumed to be onto, all the assertions are equivalent.

Definition. Let $X$ be a Banach space, $T \in L(X)$ and $\lambda \in \sigma(T)$.

- We say that $\lambda$ is an eigenvalue of $T$ if $\lambda I - T$ is not one-to-one, i.e., whenever there is $x \in X \setminus \{0\}$ such that $Tx = \lambda x$ (then $x$ is an eigenvector associated to $\lambda$). The set of all the eigenvalues is called the point spectrum of $T$ and is denoted by $\sigma_p(T)$.

- We say that $\lambda$ is an approximate eigenvalue of $T$ if there is a sequence of vectors $(x_n)$ of norm one such that $(\lambda I - T)x_n \to 0$. The set of all the approximate eigenvalues is called the approximate point spectrum of $T$ and is denoted by $\sigma_{ap}(T)$.

- We say that $\lambda$ belongs to the continuous spectrum $\sigma_c(T)$ if $\lambda I - T$ is one-to-one, has dense range but is not onto.

- We say that $\lambda$ belongs to the residual spectrum $\sigma_r(T)$ (also called compression spectrum) if $\lambda I - T$ is one to one and its range is not dense.

Proposition 2 (on subsets of the spectrum). Let $X$ be a Banach space and $T \in L(X)$. Then the following assertions hold:

(a) $\sigma_p(T) \subset \sigma_{ap}(T)$.

(b) $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$ if and only if $\lambda I - T$ is an isomorphism of $X$ into $X$.

(c) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$.

(d) $\sigma_r(T) = \sigma_{ap}(T) \setminus (\sigma_{p}(T) \cup \sigma_{r}(T)) = \sigma(T) \setminus (\sigma_{p}(T) \cup \sigma_{r}(T))$.

(e) $\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T)$ if and only if $\lambda I - T$ is an isomorphism of $X$ onto a proper closed subspace of $X$.

Definition. Let $H$ be a Hilbert space and $T \in L(H)$.

- The numerical range of $T$ is the set $W(T) = \{\langle Tx, x \rangle; x \in H, \|x\| = 1\}$.

- The numerical radius of $T$ is defined by $w(T) = \sup \{ |\lambda| ; \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle| ; x \in H, \|x\| = 1 \}$.

Lemma 3 (polarization formula for an operator). Let $H$ be a Hilbert space and $T \in L(H)$. For each $x, y \in H$ the following formula holds:

$$\langle Tx, y \rangle = \frac{1}{4} \left( \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle + i \langle T(x + iy), x + iy \rangle - i \langle T(x - iy), x - iy \rangle \right).$$
Proposition 4 (properties of the numerical radius). Let $H$ be a Hilbert space.

(a) The numerical radius $w$ is an equivalent norm on $L(H)$ satisfying $\frac{1}{2} \|T\| \leq w(T) \leq \|T\|$ for $T \in L(H)$.

(b) If $T \in L(H)$ satisfies $(Tx, x) = 0$ for all $x \in H$, then $T = 0$.

(c) If $S, T \in L(H)$ satisfy $(Tx, x) = (Sx, x)$ for all $x \in H$, then $S = T$.

(d) $T(W)$ is a connected subset of $\mathbb{C}$ for $T \in L(H)$.

(e) $\sigma_p(T) \subset W(T)$ and $\sigma(T) \subset W(T)$ for $T \in L(H)$.

(f) $w(T) \geq \|T\|$ for $T \in L(H)$.

Proposition 5 (structure of normal operators). Let $H$ be a Hilbert space and $T \in L(H)$. The operator $T$ is normal if and only if $\|Tx\| = \|T^*x\|$ for each $x \in H$. If $T$ is normal, then the following assertions hold.

(a) $\ker T = \ker T^*$ and $\ker T = (R(T)) \perp$.

(b) $R(T)$ is dense if and only if $T$ is one-to-one. Hence, $\sigma_r(T) = 0$ and $\sigma(T) = \sigma_p(T)$.

(c) If $\lambda \in \mathbb{C}$ and $x \in H$ then $Tx = \lambda x$ if and only if $T^*x = \overline{\lambda}x$. In particular, $\sigma_p(T^*) = \{\overline{\lambda}; \lambda \in \sigma_p(T)\}$.

(d) If $\lambda_1, \lambda_2 \in \sigma_p(T)$ are distinct, then $\ker(\lambda_1 I - T) \perp \ker(\lambda_2 I - T)$.

Proposition 6 (characterization of orthogonal projections). Let $H$ be a Hilbert space and let $P \in L(H)$ be a projection (i.e., $P^2 = P$). The following assertions are equivalent:

(i) $P$ is an orthogonal projection, i.e., $\ker P \perp R(P)$.

(ii) $P$ is self-adjoint.

(iii) $P$ is normal.

(iv) $\langle Px, x \rangle = \|Px\|^2$ for $x \in H$.

(v) $\langle Px, x \rangle \geq 0$ for $x \in H$.

(vi) $\|P\| \leq 1$.

Moreover, if $P, Q \in L(H)$ are two orthogonal projections, then $R(P) \perp R(Q)$ if and only if $PQ = 0$. In this case $P$ and $Q$ are called mutually orthogonal.

Proposition 7 (spectrum of a self-adjoint operator). Let $H$ be a Hilbert space and $T \in L(H)$.

(a) $T$ is self-adjoint if and only if $W(T) \subset \mathbb{R}$.

(b) Suppose that $T$ is self-adjoint and set $a = \inf W(T)$ and $b = \sup W(T)$. Then $\sigma(T) \subset [a, b]$, $a, b \in \sigma(T)$,

$$\|T\| = \max\{|a|, |b|\}$$

and $\sigma(T)$ contains one of the numbers $\|T\|, -\|T\|$.

(c) If $\lambda_1, \lambda_2 \in \sigma_p(T)$ are distinct, then $\ker(\lambda_1 I - T) \perp \ker(\lambda_2 I - T)$.

Remarks and definitions.

(1) Operators satisfying the two equivalent conditions from Proposition 7(c) are called positive.

(2) $T^*T$ is a positive operator for any $T \in L(H)$.

(3) If $T \in L(H)$, we define $|T| = \sqrt{T^*T}$ (i.e., we apply the continuous function $t \mapsto \sqrt{t}$ to the positive operator $T^*T$).

(4) If $T$ is normal, then the operator $|T|$ defined above coincides with the operator obtained by applying the continuous function $\lambda \mapsto |\lambda|$ to the operator $T$. If $T$ is not normal, then $|T| \neq |T^*|$.

(5) An operator $U \in L(H)$ is said to be a partial isometry if there is a closed subspace $H_1 \subset H$ such that $U|_{H_1}$ is an isometry of $H_1$ into $H$ and $U|_{H_2} = 0$.

Theorem 8 (polar decomposition). Let $H$ be a Hilbert space and $T \in L(H)$. Then there is a unique partial isometry $U \in L(H)$ such that $T = U|_T$ and $U = 0$ on $R(|T|)$.

Moreover, $U^*$ is also a partial isometry and $|T| = U^*T$ and $U^* = 0$ on $R(|T|)$.

Theorem 9 (Hilbert-Schmidt). Let $H$ be a Hilbert space and $T \in L(H)$ be a compact normal operator. Then there is an orthonormal basis of $H$ consisting of eigenvectors of $T$. Moreover, if $T \neq 0$, then there exist an orthonormal system $(x_k)_{k \in \mathbb{N}}$ and nonzero complex numbers $(\lambda_k)_{k \in \mathbb{N}}$, where either $N = \mathbb{N}$ or $N = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$, such that

$$Tx = \sum_{k \in \mathbb{N}} \lambda_k \langle x, x_k \rangle x_k, \quad x \in H.$$