# VII. Elements of vector integration

**Convention:** In this chapter we will use the following notation:

- $(M, \mathcal{A})$ is a fixed measurable space, i.e., $M$ is a nonempty set and $\mathcal{A}$ is a $\sigma$-algebra of subsets of $M$.
- $(\Omega, \Sigma, \mu)$ is a fixed complete measure space, i.e., $\Omega$ is a nonempty set, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mu$ is a non-negative $\sigma$-additive measure on $\Sigma$, which is moreover complete.
- $X$ is a fixed Banach space over $\mathbb{F}$.

**Remarks:**

(1) $(\Omega, \Sigma)$ is a special case of a measurable space. Therefore, whatever is below stated for $(M, \mathcal{A})$, can be applied to $(\Omega, \Sigma)$ as well.

(2) We do not a priori assume thet $\mu$ is finite or $\sigma$-finite, even though these cases are the most important ones.

## VII.1 Measurability of vector-valued functions

**Definition.** Let $f : M \to X$ be a mapping. The function $f$ is said to be

- **simple**, if its range is a finite set, i.e., if $f = \sum_{j=1}^{k} x_j \chi_{A_j}$, where $x_1, \dots, x_k \in X$ and $A_1, \dots, A_k$ are nonempty pairwise disjoint subsets of $M$;
- **simple measurable**, if it can be expressed as above and, moreover, $A_1, \dots, A_k \in \mathcal{A}$;
- (**strongly**) **$\mathcal{A}$-measurable** if there exists a sequence $(u_n)$ of simple measurable functions pointwise converging to $f$ (i.e.m such that $\lim\limits_{n\to\infty} \|u_n(t) - f(t)\| = 0$ for each $t \in M$);
- **Borel $\mathcal{A}$-measurable** if $f^{-1}(U) \in \mathcal{A}$ for each $U \subset X$ open;
- **weakly $\mathcal{A}$-measurable**, if $\varphi \circ f : M \to \mathbb{F}$ is (Borel) $\mathcal{A}$-measurable for each $\varphi \in X^*$.

**Proposition 1.**

(a) *Simple functions, simple measurable functions, strongly $\mathcal{A}$-measurable functions and weakly $\mathcal{A}$-measurable functions form vector spaces.*

(b) *Let $(f_n)$ be a sequence of functions $f_n : M \to X$ pointwise converging to a function $f : M \to X$. If all the functions $f_n$ are Borel $\mathcal{A}$-measurable (or weakly $\mathcal{A}$-measurable), the same holds for $f$.*

(c) *Let $f : M \to X$ be a function. Then*

$$f \text{ strongly } \mathcal{A}\text{-measurable} \Rightarrow f \text{ Borel } \mathcal{A}\text{-measurable} \Rightarrow f \text{ weakly } \mathcal{A}\text{-measurable}$$

*For simple functions all the mentioned types of measurability coincide.*

(d) *If $f : M \to X$ is strongly $\mathcal{A}$-measurable, then $f(M)$ is a separable subset of $X$.*

(e) *If $f : M \to X$ is Borel $\mathcal{A}$-measurable, then $\omega \mapsto \|f(\omega)\|$ is a $\mathcal{A}$-measurable (scalar-valued) function.*

**Remarks:**

(1) Borel $\mathcal{A}$-measurable functions form a vector space if $X$ is separable (by Theorem 3), in general they need not form a vector space.

(2) The converse implications in (c) fail, see Examples 6.

**Lemma 2.** *Let $(f_n)$ be a sequence of strongly $\mathcal{A}$-measurable functions $f_n : M \to X$ pointwise converging to a function $f : M \to X$. Then $f$ is strongly $\mathcal{A}$-measurable as well.*

**Theorem 3** (Pettis). *Let $f : M \to X$ be a function. The following assertions are equivalent:*

(i) *$f$ is strongly $\mathcal{A}$-measurable.*
(ii) *$f$ is Borel $\mathcal{A}$-measurable and $f(M)$ is a separable subset of $X$.*
(iii) *$f$ is weakly $\mathcal{A}$-measurable a $f(M)$ is a separable subset of $X$.*

**Definition.** Let $f : \Omega \to X$ be a mapping. The function $f$ is said to be

- **(strongly) $\mu$-measurable** if there exists a sequence $(u_n)$ of simple measurable functions $u_n : \Omega \to X$ almost everywhere converging to $f$ (i.e. such that $\lim_{n\to\infty} \|u_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$);
- **Borel $\mu$-measurable** (or **weakly $\mu$-measurable**), it it is Borel $\Sigma$-measurable (or weakly $\Sigma$-measurable).

**Remarks:**

(1) Let $f : \Omega \to X$ be a function. Then

$$f \text{ strongly } \mu\text{-measurable } \Rightarrow f \text{ Borel } \mu\text{-measurable } \Rightarrow f \text{ weakly } \mu\text{-measurable}$$

(2) If $f : \Omega \to X$ is (strongly) $\mu$-measurable, then

$$\exists Y \subset\subset X \text{ separable } \exists N \in \Sigma : \mu(N) = 0 \;\&\; f(\Omega \setminus N) \subset Y.$$

A function satisfying this condition is called **essentially separably valued**.

**Lemma 4.** *Let $(f_n)$ be a sequence of strongly $\mu$-measurable functions $f_n : M \to X$ almost everywhere converging to a function $f : M \to X$. Then $f$ is strongly $\mu$-measurable as well.*

**Theorem 5** (Pettis). *Let $f : \Omega \to X$ be a function. The following assertions are equivalent:*

(i) *$f$ is strongly $\mu$-measurable.*
(ii) *$f$ is Borel $\mu$-measurable and essentially separably valued.*
(iii) *$f$ is weakly $\mu$-measurable and essentially separably valued.*

**Examples 6.**

(1) *Let $\Omega = [0, 1]$, let $\mu$ be the Lebesgue measure on $[0, 1]$ and let $\Sigma$ be the $\sigma$-algebra of all the Lebesgue measurable subsets of $[0, 1]$. Consider the function $f : [0, 1] \to \ell^2([0, 1])$ defined by $f(t) = \boldsymbol{e}_t$, $t \in [0, 1]$, where $\boldsymbol{e}_t$ denotes the respective canonical unit vector.*

*Then $f$ is weakly $\mu$-measurable, but fails to be essentially separably valued, hence it is not strongly $\mu$-measurable. It is neither Borel $\mu$-measurable.*

(2) *Let $(\Omega, \Sigma, \mu)$ and $f$ be as in (1). Let moreover $h : [0, 1] \to [0, \infty)$ be any function. Then the function $h \cdot f$ is weakly $\mu$-measurable as well. Furher, for $t \in [0, 1]$ one has $\|h(t)f(t)\| = h(t)$. Therefore, if we choose $h$ to be non-measurable, then $g = h \cdot f$ is weakly $\mu$-measurable, but the function $t \mapsto \|g(t)\|$ is not measurable.*

(3) *Let $\Omega = [0, 1]$, let $\Sigma$ be the $\sigma$-algebra of all the subsets of $[0, 1]$, let $\mu$ be the counting measure and let $f$ be as in (1). Then $f$ is Borel $\mu$-measurable, but fails to be essentially separably valued, thus it is not strongly $\mu$-measurable.*

**Remark:** The question, whether for a finite measure $\mu$ any Borel $\mu$-measurable function is essentially separably valued (and hence strongly $\mu$-measurable), is more complicated. The answer depends on additional axioms of the set theory.