

VII.2 Integrability of vector-valued functions

Definition.

- Let $f : \Omega \rightarrow X$ be a simple measurable function of the form $f = \sum_{j=1}^k x_j \chi_{E_j}$ (where $E_1, \dots, E_k \in \Sigma$ are pairwise disjoint and $x_1, \dots, x_k \in X$). Let $E \in \Sigma$. We say that f is **integrable over E** , if for each $j \in \{1, \dots, k\}$ one has $\mu(E \cap E_j) < \infty$ or $x_j = \mathbf{o}$. By the **integral of f over E** we mean the element of X defined by the formula

$$\int_E f \, d\mu = \sum_{j=1}^k \mu(E \cap E_j) x_j,$$

where by convention $\infty \cdot \mathbf{o} = \mathbf{o}$. If f is integrable over Ω , it is called **integrable**.

- Let $f : \Omega \rightarrow X$ be strongly μ -measurable. The function f is said to be **Bochner integrable** if there exists a sequence (f_n) of simple integrable functions such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0,$$

where the integral is in the Lebesgue sense. By the **Bochner integral** of f we then mean the element of X defined by

$$(B) \int_{\Omega} f \, d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, d\mu.$$

- A function $f : \Omega \rightarrow X$ is said to be **weakly integrable** if $\varphi \circ f$ is integrable (i.e., $\varphi \circ f \in L^1(\mu)$) for each $\varphi \in X^*$.

Proposition 7 (basic properties of the Bochner integral).

- Integrable simple functions form a vector space; and the mapping assigning to a simple integrable function f its integral $\int_{\Omega} f \, d\mu$ is linear.*
- Let f be a simple measurable function. Then f is integrable if and only if the function $\omega \mapsto \|f(\omega)\|$ is integrable. In this case*

$$\left\| \int_{\Omega} f \, d\mu \right\| \leq \int_{\Omega} \|f(\omega)\| \, d\mu(\omega).$$

- The limit defining the Bochner integral does exist and does not depend on the choice of the sequence (f_n) .*
- Bochner integrable functions form a vector space; and the mapping assigning to a Bochner integrable function its Bochner integral is linear.*
- If $f : \Omega \rightarrow X$ is Bochner integrable, then the function $\omega \mapsto \|f(\omega)\|$ is integrable and*

$$\left\| (B) \int_{\Omega} f \, d\mu \right\| \leq \int_{\Omega} \|f(\omega)\| \, d\mu(\omega).$$

- If $f : \Omega \rightarrow X$ Bochner integrable, then $\chi_E \cdot f$ is Bochner integrable for each $E \in \Sigma$. (The value $(B) \int_{\Omega} \chi_E \cdot f \, d\mu$ is called the **Bochner integral of f over E** and it is denoted by $(B) \int_E f \, d\mu$.)*

Theorem 8 (a characterization of Bochner integrability). *Let $f : \Omega \rightarrow X$ be a strongly μ -measurable function. Then f is Bochner integrable if and only if $\int_{\Omega} \|f(\omega)\| \, d\mu(\omega) < \infty$.*

Theorem 9 (Lebesgue dominated convergence theorem for Bochner integral). *Let (f_n) be a sequence of Bochner integrable functions $f_n : \Omega \rightarrow X$ almost everywhere converging to a function $f : \Omega \rightarrow X$. Let $g : \Omega \rightarrow \mathbb{R}$ be an integrable function such that for each $n \in N$ one has $\|f_n(\omega)\| \leq g(\omega)$ for almost all $\omega \in \Omega$. Then f is Bochner integrable and $(B) \int_{\Omega} f \, d\mu = \lim_{n \rightarrow \infty} (B) \int_{\Omega} f_n \, d\mu$.*

Proposition 10 (absolute continuity of Bochner integral). *Let $f : \Omega \rightarrow X$ be Bochner integrable. Then:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall E \in \Sigma : \mu(E) < \delta \Rightarrow \left\| \int_E f \, d\mu \right\| < \varepsilon.$$

Proposition 11 (weak integral). *Let $f : \Omega \rightarrow X$ be weakly integrable. Then the mapping*

$$F(\varphi) = \int_{\Omega} \varphi \circ f \, d\mu, \quad \varphi \in X^*,$$

is a continuous linear functional on X^ , i.e., $F \in X^{**}$.*

Definition, notation and remarks:

- (1) The element $F \in X^{**}$ provided by Proposition 11 is called the **weak integral** (or the **Dunford integral**) of f , it is denoted by $(D) \int_{\Omega} f \, d\mu$.
- (2) Let $f : \Omega \rightarrow X$ be weakly integrable. Then $\chi_E \cdot f$ is weakly integrable for each $E \in \Sigma$. The respective weak integral is denoted by $(D) \int_E f \, d\mu$.
- (3) We say that $f : \Omega \rightarrow X$ is **Pettis integrable** if
 - f is weakly integrable and, moreover,
 - the weak integral $(D) \int_E f \, d\mu$ belongs to $\varkappa(X)$ (where $\varkappa : X \rightarrow X^{**}$ is the canonical embedding) for each $E \in \Sigma$.

The Pettis integral of f over E is then the respective element of X and it is denoted by $(P) \int_E f \, d\mu$. I.e., for $x \in X$ then one has

$$x = (P) \int_E f \, d\mu \Leftrightarrow \forall \varphi \in X^* : \varphi(x) = \int_E \varphi \circ f \, d\mu.$$

Remarks:

- (1) In order that f is Pettis integrable, $(D) \int_E f \, d\mu \in \varkappa(X)$ should hold for each $E \in \Sigma$. It is not enough if it is satisfied in case $E = \Omega$.
- (2) A weakly integrable function need not be Pettis integrable.
- (3) A Pettis integrable function need not be strongly μ -measurable. For example, the function from Example 6(1) is Pettis integrable, its integral is zero, but it is not essentially separably valued.
- (4) Any Bochner integrable function is Pettis integrable (this follows from Proposition 12), the converse implication fails even for pro strongly μ -measurable functions (see Example 13).

Proposition 12 (Bochner integral and a bounded operator). *Let $f : \Omega \rightarrow X$ be Bochner integrable, let Y be a Banach space and let $L : X \rightarrow Y$ be a bounded linear operator. Then $L \circ f$ is Bochner integrable and*

$$(B) \int_{\Omega} L \circ f \, d\mu = L \left((B) \int_{\Omega} f \, d\mu \right).$$

Remark: The preceding proposition shows that the Bochner integrability implies the Pettis one and, moreover, it can be used to compute the Bochner integral of a function: To this end it is necessary to show that the Bochner integral exists, its value can be then computed using suitable functionals or operators.

Example 13. *Let $\Omega = \mathbb{N}$, let Σ be the σ -algebra of all the subsets of \mathbb{N} , let μ be the counting measure and let $f : \Omega \rightarrow X$. Then:*

- (a) f is Bochner integrable if and only if the series $\sum_{n \in \mathbb{N}} f(n)$ is absolutely convergent. The Bochner integral then equals the sum of the series.
- (b) If the series $\sum_{n \in \mathbb{N}} f(n)$ is unconditionally convergent, then f is Pettis integrable and its Pettis integral equals the sum of the series.

Remark: In Example 13(b) the converse implication holds as well – if f is Pettis integrable, then the series $\sum_{n \in \mathbb{N}} f(n)$ is unconditionally convergent. The proof is more complicated, this statement is the content of Orlicz-Pettis theorem.