

FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2016/2017

PROBLEMS TO CHAPTER V

PROBLEMS TO SECTION V.1 – LINEAR TOPOLOGIES AND THEIR GENERATION

Problem 1. Let X be a vector space and let $A \subset X$ be a nonempty set. Show that A is convex if and only if $(\alpha + \beta)A = \alpha A + \beta A$ for each $\alpha, \beta > 0$.

Problem 2. Let X be a vector space and let \mathcal{T} be a topology on X .

- (1) Show that \mathcal{T} is translation invariant if and only if the addition on X is separately continuous (i.e. $x \mapsto x + y$ is continuous for each $y \in X$).
- (2) Find X and \mathcal{T} such that the addition is continuous (in both variables simultaneously, as a mapping $X \times X \rightarrow X$) but the multiplication is not continuous.
- (3) Let \mathcal{T} be translation invariant and let there exist a base of neighborhoods of zero consisting of convex sets. Show that the addition is continuous (in both variables).
- (4) Find a topology on \mathbb{R}^2 , in which the addition is separately continuous but not continuous.

Hint: (2) Consider the discrete topology. (4) Consider, for example, a neighborhood base of zero made by sets $\{(0, 0)\} \cup \{(x, y); |y| < |x| < r\}$, $r > 0$.

Problem 3. Let X be a vector space and let ρ be a translation invariant metric on X . Show that ρ generates a linear topology on X if and only if

$$\lim_{x \rightarrow \mathbf{o}} \sup_{\lambda \in \mathbb{F}, |\lambda| \leq 1} \rho(\lambda x, 0) = 0.$$

Find a translation invariant metric failing this condition.

Problem 4. Let X be a vector space. Let \mathcal{U} be the family of all the absolutely convex absorbing subsets of X .

- (1) Show that \mathcal{U} is a base of neighborhoods of zero in a Hausdorff locally convex topology \mathcal{T} on X .
- (2) Show that this topology \mathcal{T} is the strongest locally convex topology on X .
- (3) Show that any convergent sequence in (X, \mathcal{T}) is contained in a finite-dimensional subspace.

Hint: (3) Suppose it is not the case. Then there exists a linearly independent sequence (x_n) which converges to zero. Complete this sequence to an algebraic basis of X . Describe an absolutely convex absorbing set, not containing any of the vectors x_n .

Problem 5. (1) Let $A \subset \mathbb{R}^2$ be balanced and absorbing. Show that $A + A$ is a neighborhood of zero (in the standard topology).
(2) Find a balanced absorbing $A \subset \mathbb{R}^2$, which is not a neighborhood of zero.
(3) Using the previous two points deduce that the family of all the balanced absorbing subsets of \mathbb{R}^2 is not the base of neighborhoods of zero in any linear topology on \mathbb{R}^2 .

Hint: (1) Show that $A + A$ contains a rectangle of the form $[-a, a] \times [-b, b]$.

Problem 6. (1) Show that the convex hull of a balanced subset of a vector space is again balanced, and hence absolutely convex.

(2) Show that the balanced hull of a convex set need not be convex.

Hint: (2) Consider a suitable segment in \mathbb{R}^2 .

Problem 7. Let X be the space of all the Lebesgue measurable functions on $[0, 1]$ (with values in \mathbb{F} ; we identify the functions, which are equal almost everywhere). For $f, g \in X$ set

$$\rho(f, g) = \int_0^1 \min\{1, |f - g|\}.$$

(1) Show that ρ is a metric generating a linear topology on X .

(2) Show that the convergence of sequences in the metric ρ coincide with the convergence in measure.

(3) Is the resulting topology locally convex?

Hint: (1) Show that the operations are continuous using the convergence of sequences. (3) Show that for each $r > 0$ the convex hull of the set $\{f \in X; \rho(f, 0) < r\}$ is the whole X .

Problem 8. Let X be a TVS and let $A \subset X$ be a balanced set with nonempty interior.

(1) Show that $\text{int } A$ is balanced if and only if $0 \in \text{int } A$.

(2) Show on a counterexample that $\text{int } A$ need not be balanced.

Problem 9. Let (X, \mathcal{T}) be a TVS and let $A \subset X$ be nonempty. Show that

$$\overline{A} = \bigcap \{A + U; U \in \mathcal{T}(0)\}.$$

Problem 10. Let (X, \mathcal{T}) be a non-Hausdorff TVS.

- Denote $Z = \overline{\{0\}} = \bigcap \mathcal{T}(0)$. Show that Z is a vector subspace of X .
- Let $Y = X/Z$ be the quotient vector space and let $q : X \rightarrow Y$ be the canonical quotient mapping. Let \mathcal{R} be the quotient topology on Y (i.e., $\mathcal{R} = \{U \subset Y; q^{-1}(U) \in \mathcal{T}\}$). Show that (Y, \mathcal{R}) is a HTVS.
- Show that (Y, \mathcal{R}) is locally convex if and only if (X, \mathcal{T}) is locally convex.

PROBLEMS TO SECTION V.2 – BOUNDED SETS, CONTINUOUS LINEAR MAPPINGS

Problem 11. Let X be a TVS and let $A \subset X$. Show that A is bounded if and only if each countable subset of A is bounded.

Problem 12. Let X be a TVS and let $A, B \subset X$ be bounded sets. Show that the sets $A \cup B$, $A + B$, \overline{A} , $b(A)$ are bounded as well.

Problem 13. Let X be a LCS and let $A \subset X$ be a bounded set. Show that the sets $\text{co } A$ and $\text{aco } A$ are bounded as well.

Problem 14. Let $X = L^p([0, 1])$ where $p \in (0, 1)$. Show that $A = \{f \in X; \|f\|_p < 1\}$ is a bounded set whose convex hull is not bounded.

Hint: Show that $\text{co } A = X$.

Problem 15. Let X be a normed linear space and $A \subset X$. Show that A is bounded as a subspace of the TVS X if and only if it is bounded in the metric generated by the norm.

Problem 16. Let X be a TVS whose topology is generated by a translation invariant metric ρ .

- (1) Show that any set $A \subset X$ bounded in X is bounded in the metric ρ as well.
- (2) Show that a set $A \subset X$ bounded in the metric ρ need not be bounded in the TVS X .

Hint: (2) The metric ρ itself may be bounded.

Problem 17. Consider the space of test functions $\mathcal{D}(\Omega)$ with the topology from Example V.5(2). Let $\Lambda : \mathcal{D}(\Omega) \rightarrow \mathbb{F}$ be a linear functional. Show that $\Lambda \in \mathcal{D}'(\Omega)$ if and only if Λ is continuous.

Problem 18. Consider the space (X, \mathcal{T}) from Problem 4. Show that any linear functional $L : X \rightarrow \mathbb{F}$ is continuous.

Problem 19. Let $X = \mathbb{F}^\Gamma$ and let $A \subset X$. Show that A is bounded in X if and only if it is „pointwise bounded“, i.e., if and only if the set $\{x(\gamma); x \in A\}$ is bounded in \mathbb{F} for each $\gamma \in \Gamma$.

Problem 20. Let $X = \mathcal{C}([0, 1])$ equipped with the topology of pointwise convergence, let $Y = \mathcal{C}([0, 1])$ equipped with the topology generated by the metric ρ from Problem 7 and let $L : X \rightarrow Y$ be the identity mapping.

- (1) Show that L maps bounded sets to bounded sets.
- (2) Show that L is sequentially continuous (i.e., $f_n \rightarrow f$ in X implies $Lf_n \rightarrow Lf$ in Y).
- (3) Show that L is not continuous.

Hint: (1) Let $A \subset X$ be bounded. For $n \in \mathbb{N}$ set $F_n = \{x \in [0, 1]; \forall f \in A : |f(x)| \leq n\}$. Show that F_n are closed subsets of $[0, 1]$, whose union is the whole interval $[0, 1]$. For $\varepsilon > 0$ choose $n \in \mathbb{N}$ such that $\lambda([0, 1] \setminus F_n) < \frac{\varepsilon}{2}$; and $m > n$ such that $\frac{n}{m} < \frac{\varepsilon}{2}$. Show that then $L(A) \subset m\{g \in Y; \rho(g, 0) < \varepsilon\}$. Deduce the boundedness of $L(A)$ in Y . (2) Use Lebesgue dominated convergence theorem. (3) Let $U = \{f \in Y; \rho(f, 0) < \frac{1}{2}\}$. Then U is a neighborhood of zero in Y . Show that $L^{-1}(U)$ is not a neighborhood of zero in X . (For any finite $F \subset [0, 1]$ find $f \in \mathcal{C}([0, 1])$ such that $f|_F = 0$ but $f = 1$ on a set of measure at least $\frac{1}{2}$.)

Problem 21. Let X be a TVS and let (x_n) be a sequence of elements of X . Show that the sequence (x_n) is bounded in X if and only if for any sequence (λ_n) in \mathbb{F} one has $\lambda_n \rightarrow 0 \Rightarrow \lambda_n x_n \rightarrow \mathbf{o}$.

Problem 22. Let X be a metrizable TVS and let (x_n) be a sequence of elements of X . Show that there exists a sequence of strictly positive numbers (λ_n) such that $\lambda_n x_n \rightarrow \mathbf{o}$.

Hint: Let ρ be a metric generating the topology on X . Show and then use that $\lim_{t \rightarrow 0^+} \rho(\mathbf{o}, tx) = 0$ for each $x \in X$.

Problem 23. Is the assertion from Problem 22 true also for non-metrizable TVS?

Hint: Consider the space from Problem 4.

Problem 24. Let X be a TVS, whose topology is generated by a translation invariant metric ρ . Let (x_n) be a sequence of elements X converging to zero. Show that there exists a sequence of positive numbers (λ_n) such that $\lambda_n \rightarrow \infty$ and $\lambda_n x_n \rightarrow \mathbf{o}$.

Hint: By the translation invariance of ρ it follows $\rho(\mathbf{o}, nx) \leq n\rho(\mathbf{o}, x)$ for $x \in X$ and $n \in \mathbb{N}$.

Problem 25. Is the assertion from the previous problem valid for a general TVS?

Hint: Consider, e.g., $X = c_0$ or $X = \ell^p$ for $p \in (1, \infty)$ with the weak topology (see Section VI.1), let (x_n) be the sequence of canonical unit vectors.

PROBLEMS TO SECTION V.3 – FINITE-DIMENSIONAL AND INFINITE-DIMENSIONAL SPACES

Problem 26. Show that any finite-dimensional TVS is locally convex.

Hint: For a Hausdorff space use Proposition V.9. For the general case use Problem 10.

Problem 27. Let X be a vector space of infinite dimension. Show that on X there exists a Hausdorff linear topology, which is not locally convex.

Hint: Let $(e_j)_{j \in J}$ be an algebraic basis of X . Choose $p \in (0, 1)$ and define a metric on X by $\rho(\sum a_j e_j, \sum b_j e_j) = \sum |a_j - b_j|^p$.

Problem 28. Let X be a metrizable TVS of infinite dimension. Show that there exists a discontinuous linear functional on X .

Hint: Use an algebraic basis of X and Problem 22.

Problem 29. Is there a discontinuous linear functional on each infinite-dimensional HTVS (or HLCS)?

Hint: Use Problem 18.

PROBLEMS TO SECTION V.4 – METRIZABILITY OF TVS

Problem 30. Let X be a vector space over \mathbb{F} and let $p : X \rightarrow [0, \infty)$ be an F -norm on X .

- (1) Show that the formula $\rho(x, y) = p(x - y)$ defines a translation invariant metric on X , which generates a linear topology on X .
- (2) Show that the family $\left\{ \left\{ x \in X; p(x) < \frac{1}{n} \right\}; n \in \mathbb{N} \right\}$ is a base of neighborhoods of zero in this topology.

Problem 31. Let X be a vector space over \mathbb{F} and let $q : X \rightarrow [0, \infty)$ be a **quasinorm** on X , i.e., a mapping with the following properties:

- $\forall x \in X : q(x) = 0 \Leftrightarrow x = \mathbf{o}$;
- $\forall x \in X \forall \lambda \in \mathbb{F} : q(\lambda x) = |\lambda| q(x)$;
- $\exists C \geq 1 \forall x, y \in X : q(x + y) \leq C(q(x) + q(y))$.

- (1) Show that the family $\left\{ \left\{ x \in X; q(x) < r \right\}; r \in (0, \infty) \right\}$ is a base of neighborhoods of zero in a linear topology on X .
- (2) Show that this topology is metrizable.
- (3) Show that $x_n \rightarrow x$ in this topology if and only if $q(x_n - x) \rightarrow 0$.

Hint: (2) Show that there exists a countable base of neighborhoods of zero.

Problem 32. Let X be a vector space over \mathbb{F} , let $q : X \rightarrow [0, \infty)$ be an F -norm on X and let $p \in (0, 1)$. We say that q is a **p -norm**, if $q(\lambda x) = |\lambda|^p q(x)$ for each $x \in X$ and $\lambda \in \mathbb{F}$.

- (1) Show that the function $f \mapsto \|f\|_p$ is a p -norm on $L^p(\mu)$.

- (2) Show that the function $f \mapsto (\|f\|_p)^{1/p}$ is a quasinorm on $L^p(\mu)$.
- (3) Let q be a p -norm on X . Show that $q^{1/p}$ is a quasinorm on X and estimate C .

Hint: (2) is a special case of (3). Set $\alpha = \frac{1}{p}$. By analysing monotonicity of a suitable function show that $a^\alpha + b^\alpha \leq (a+b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$ whenever $a, b \geq 0$.

Problem 33. Let X be a TVS and let p be an F -seminorm on X . Show that p is continuous if and only if it is continuous at zero.

Problem 34. Let X be a TVS and let U be a neighborhood of zero. Show that there exists a continuous F -seminorm p on X such that $\{x \in X; p(x) < 1\} \subset U$.

Hint: Let (V_n) be sequence of balanced neighborhoods of zero satisfying $V_1 + V_1 \subset U$ and $V_{n+1} + V_{n+1} \subset V_n$ for $n \in \mathbb{N}$. Apply the construction from the proof of Proposition V.14 to this sequence.

Problem 35. Using the previous problem show that any TVS is completely regular.

PROBLEMS TO SECTION V.5 – SEMINORMS, MINKOWSKI FUNCTIONALS,
F-SEMINORMS

Problem 36. On a counterexample show that the Minkowski functional of a balanced neighborhood of zero need not be continuous.

Hint: It may happen that there is $x \in X$ and numbers $0 < a < b$ such that the segment $\{tx; t \in [a, b]\}$ is contained in the boundary of a given balanced neighborhood of zero. A counterexample may be constructed already in \mathbb{R}^2 .

Problem 37. Let X be a TVS, let $A \subset X$ be a balanced neighborhood of zero and p_A its Minkowski functional. Show that the following three conditions are equivalent:

- (i) p_A is continuous on X ;
- (ii) For each $x \in \bar{A}$ one has $\{tx; t \in [0, 1]\} \subset \text{int } A$;
- (iii) $\text{int } A = \{x \in X; p_A(x) < 1\}$ & $\bar{A} = \{x \in X; p_A(x) \leq 1\}$.

Problem 38. Let X be a TVS, let $A \subset X$ be a subset containing the origin and let $p \in (0, 1)$. We say that the set A is p -convex if

$$\forall x, y \in A \forall s, t \in [0, 1] : s^p + t^p = 1 \Rightarrow sx + ty \in A.$$

- (1) Let A be p -convex, let $x_1, \dots, x_n \in A$ and let $t_1, \dots, t_n \in [0, 1]$ satisfy $t_1^p + \dots + t_n^p = 1$. Show that $t_1x_1 + \dots + t_nx_n \in A$.
- (2) Let $0 < p < q < 1$. Show that

$$A \text{ convex} \Rightarrow A \text{ } q\text{-convex} \Rightarrow A \text{ } p\text{-convex}.$$

- (3) On an example demonstrate that a p -convex set need not be convex.
- (4) Let A be a p -convex neighborhood of zero. Show that its Minkowski functional is continuous.

Hint: (1) Use the mathematical induction. (2) Use that $0 \in A$. (3) Consider for example the set $\{f \in L^p([0, 1]); \|f\|_p < 1\}$. (4) Use the characterization from the previous problem.

Problem 39. Show that the topology from Problem 4 is generated by the family of all the seminorms on X .

Problem 40. Let X be a vector space and let \mathcal{P} be a family of F -seminorms on X .

(1) Show that the family

$$\left\{ \{x \in X; p_1(x) < c_1, \dots, p_k(x) < c_k\}; p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0 \right\}$$

forms a base of neighborhoods of zero in a linear topology on X .

(2) Show that the F -seminorms from \mathcal{P} are continuous in this topology.

(3) Show that this topology is Hausdorff if and only if for each $x \in X \setminus \{0\}$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$.

Problem 41. Let (X, \mathcal{T}) be a TVS. Let \mathcal{P} be the family of all the continuous F -seminorms on X . Show that the topology generated by the family \mathcal{P} in the sense of Problem 40 equals \mathcal{T} .

Hint: Use Problem 34.

Problem 42. Let X be a vector space. Show that on X there exists the strongest linear topology and that this topology is Hausdorff.

Hint: Apply the construction from Problem 40 to the family of all the F -seminorms on X .

Problem 43. Let X be a vector space of countable algebraic dimension. Show that the strongest linear topology on X is locally convex.

Hint: Let \mathcal{T} be the topology from Problem 4. Show that each F -seminorm on X is continuous in \mathcal{T} : Let p be an F -seminorm on X and let (e_n) be an algebraic basis. For each $\varepsilon > 0$ show that there is (t_n) , a sequence of strictly positive numbers such that $\text{aco}\{t_n e_n; n \in \mathbb{N}\} \subset \{x; p(x) < \varepsilon\}$.

Problem 44. Let X be vector space of uncountable algebraic dimension. Show that the strongest linear topology on X is not locally convex.

Hint: Show that the F -seminorm defined as in the Problem 27 is not continuous in the topology from Problem 4.

PROBLEMS TO SECTION V.6 - F -SPACES, FRÉCHET SPACES, TOTALLY BOUNDED SETS

Problem 45. Let X be a TVS and let $A, B \subset X$ be totally bounded subsets. Show that the sets $A \cup B$, $A + B$, \overline{A} , $b(A)$ are totally bounded as well.

Problem 46. Let X be a TVS, whose topology is generated by a translation invariant metric ρ . Show that a set $A \subset X$ is totally bounded in the TVS X if and only if it is totally bounded in the metric ρ .

Problem 47. On a counterexample show that in an F -space which is not locally convex, the closed convex hull of a compact set need not be compact.

Hint: A counterexample may be constructed for example in $L^p([0, 1])$ for $p \in (0, 1)$: Choose strictly positive numbers ε , η and δ such that $p < \frac{1}{1+\varepsilon}$, $\frac{\eta}{\varepsilon} < p$, $\delta < \varepsilon$ and $\frac{\eta}{\varepsilon-\delta} < p$. For $n \in \mathbb{N}$ set $x_n = \frac{1}{n^{1+\eta}}$, $f_n = n^{1+\varepsilon} \chi_{(x_{n+1}, x_n)}$ and $t_n = \frac{1}{n^{1+\delta}}$. Show that $f_n \rightarrow 0$ in $L^p([0, 1])$, and so $\{0, f_1, f_2, f_3, \dots\}$ is a compact set. Further show, using the elements $\frac{t_1 f_1 + \dots + t_n f_n}{t_1 + \dots + t_n}$, that the convex hull of the mentioned compact set is an unbounded set in $L^p([0, 1])$.

PROBLEMS TO SECTION V.7 – SEPARATION THEOREMS

Problem 48. Let $X = L^p([0, 1])$ where $p \in (0, 1)$. Show that $X^* = \{0\}$.

Hint: Show that the only two convex open sets in X are \emptyset and X .

Problem 49. Let $X = \ell^p$ where $p \in (0, 1)$. Show that for each sequence $\mathbf{x} = (x_n) \in \ell^\infty$ the formula

$$\varphi_{\mathbf{x}}(\mathbf{y}) = \sum_{n=1}^{\infty} x_n y_n, \quad \mathbf{y} = (y_n) \in \ell^p,$$

defines a continuous linear functional on ℓ^p . Show that the mapping $\mathbf{x} \mapsto \varphi_{\mathbf{x}}$ is a linear bijection of ℓ^∞ onto X^* .

Problem 50. Let $p \in (0, 1)$. Show that ℓ^p is isomorphic (even linearly isometric) to a subspace of $L^p([0, 1])$. Using the two previous problems demonstrate on a counterexample that a continuous linear functional on a subspace of a TVS need not admit a continuous linear extension to the whole space.

Problem 51. Let X be a normed linear space of infinite dimension. Show that in X there exist two disjoint convex sets which are dense in X (and hence they cannot be separated by a nonzero element of X^*).

Hint: Use the existence of a discontinuous linear functional.

Problem 52. Let $X = \mathcal{C}([0, 1])$ be equipped with the L^2 -norm (i.e., $\|f\| = \left(\int_0^1 |f|^2\right)^{1/2}$). For $\alpha \in \mathbb{R}$ define $Y_\alpha = \{f \in X; f(0) = \alpha\}$. Show that $(Y_\alpha; \alpha \in \mathbb{R})$ is a pairwise disjoint family of dense convex sets. Show that for $\alpha \neq \beta$ the sets Y_α and Y_β cannot be separated by a nonzero element of X^* .

Problem 53. Let $X = c_0$ or $X = \ell^p$ for some $p \in [1, \infty)$ (consider the real version of these spaces). Let $\mathbf{x} = (x_n) \in X$ be an element with all the coordinates strictly positive and let $\mathbf{y} = \left(\frac{x_n}{n}\right) \in X$. Set

$$A = \{\mathbf{z} = (z_n) \in X; \forall n \in \mathbb{N} : z_n \geq 0\}, \quad B = \{-\mathbf{x} + t\mathbf{y}; t \in \mathbb{R}\}.$$

Show that A and B are disjoint closed subsets of X , which cannot be separated by a nonzero element of X^* .

Hint: Proceed by contradiction: Let $f \in X^ \setminus \{0\}$ satisfy $\sup f(B) \leq \inf f(A)$. Show that necessarily $f \geq 0$ on A and $\inf f(A) = 0$. The functional f can be represented by an appropriate sequence (by an element of ℓ^1 or ℓ^q where $\frac{1}{p} + \frac{1}{q} = 1$), show that all the entries of this sequence have to be non-negative. By the assumption $\inf f(B) \leq 0$ deduce $f(\mathbf{y}) = 0$, hence $f = 0$, a contradiction.*