

Theorem Let A be a Banach algebra. Then $\forall a \in A : \sigma(a)$
 is a nonempty compact subset of \mathbb{C}

Proof: (1) A unital $\Rightarrow \sigma(a) = \mathbb{C} \setminus \rho(a) \Rightarrow$ it is closed by Prop. 8(ii)

and by Prop. 8(v) $\sigma(a) \subset \overline{U(0, \|a\|)}$

$\Rightarrow \sigma(a)$ is bdd, hence compact

(2) A not unital $\Rightarrow \sigma(a) = \sigma_{A+}(a, 0)$ by definition, so
 it is a compact set by (1)

(3) $\sigma(a) \neq \emptyset$: A not unital $\Rightarrow 0 \in \sigma(a)$

A unital. Suppose $\sigma(a) = \emptyset$, i.e. $\rho(a) = \mathbb{C}$

Then $\forall \varphi \in A^* : \lambda \mapsto \varphi(R(\lambda, a))$ is an entire function
 (holomorphic in \mathbb{C})

Moreover, for $|\lambda| > \|a\|$ we have (by Prop. 8(v))

$$\begin{aligned} \|R(\lambda, a)\| &= \left\| \sum_{h=0}^{\infty} \frac{a^h}{\lambda^{h+1}} \right\| \leq \sum_{h=0}^{\infty} \left\| \frac{a^h}{\lambda^{h+1}} \right\| \leq \sum_{h=0}^{\infty} \frac{\|a\|^h}{|\lambda|^{h+1}} = \\ &= \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|a\|}{|\lambda|}} = \frac{1}{|\lambda| - \|a\|} \rightarrow 0 \text{ for } \lambda \rightarrow \infty \end{aligned}$$

Hence, given $\varphi \in A^* : \lim_{\lambda \rightarrow \infty} \varphi(R(\lambda, a)) = 0$

Hence, by a consequence of the Liouville theorem
 $\varphi(R(\lambda, a)) = 0$ for $\lambda \in \mathbb{C}$

Since $\varphi \in A^*$ is arbitrary, a consequence to H-B theorem
 yields $R(\lambda, a) = 0$ for $\lambda \in \mathbb{C}$.

It is a contradiction, as $(\lambda e - a)^{-1}$ cannot be 0.

Remark: $G \subset \mathbb{C}$ open $\Rightarrow \{a \in A; \sigma(a) \subset G\}$ is open in A

• WLOG A unital

• Let $\sigma(a) \subset G$ Then $r := \text{dist}(\sigma(a), \mathbb{C} \setminus G) > 0$

Let $K := \overline{U(0, r + \|a\|)} \setminus U(\sigma(a), r)$ *

Then K is a compact subset of $\mathbb{C} \setminus \sigma(a)$, $K \neq \emptyset$

($K \supset \{\lambda; |\lambda| = r + \|a\|\}$ as $U(\sigma(a), r) \subset U(0, r + \|a\|)$)

$M := \min_{\lambda \in K} \|(\lambda e - a)^{-1}\|$ (cts function on a compact set)

Let now $\|b - a\| < \min\{r, \frac{1}{M}\}$

Then: • $\|b - a\| < r \Rightarrow \|b\| < r + \|a\| \Rightarrow \sigma(b) \subset U(0, r + \|a\|)$

• $\lambda \in K \Rightarrow \|(\lambda e - b) - (\lambda e - a)\| = \|b - a\| <$

$< \frac{1}{M} \leq \frac{1}{\|(\lambda e - a)^{-1}\|}$. Hence $\lambda e - b$ is invertible
(cf. Lem. 6 (b))

So, $\lambda \in \rho(b)$

It follows that $\sigma(b) \cap K = \emptyset$

So, $\sigma(b) \subset U(0, r + \|a\|) \setminus K = U(\sigma(a), r) \subset G$

Gelfand-Mazur theorem

A unital Banach algebra

A is a field (i.e. $G(A) = A \setminus \{0\}$) $\Leftrightarrow A \cong \mathbb{C}$

Proof: \Leftarrow clear

\Rightarrow : $x \in A \Rightarrow \sigma(x) \neq \emptyset$ by Thm 9

Fix $\lambda \in \sigma(x)$. Then $\lambda e - x$ is not invertible, so by the assumption $\lambda e - x = 0$, hence $x = \lambda e$

Therefore, $T: \mathbb{C} \rightarrow A$ defined by $T(\lambda) = \lambda e$ is onto.

It is clear that T is an isometric isomorphism of Banach algebras.

Lemma 11 (on spectrum and polynomials)

A unital Banach algebra, $p(\lambda) = \sum_{j=0}^n d_j \lambda^j$ a polynomial
 $a \in A$, $p(a) := \sum_{j=0}^n d_j a^j$

(a) $p(a) \in G(A) \Leftrightarrow$ the roots of $p \subset \rho(a)$

• p constant \Rightarrow $\begin{cases} p \neq 0 \Rightarrow p$ has no roots $\subset p(a) \neq 0 \in G(A) \\ p = 0 \Rightarrow p(a) = 0 \notin G(A)$, roots of $p = \mathbb{C} \not\subset \rho(a)$ by Thm 9

• $\deg p \geq 1$ ($\deg p = n$)

$\Rightarrow p(\lambda) = d_n (\lambda - \xi_1) \cdots (\lambda - \xi_n)$, where ξ_1, \dots, ξ_n are roots

Then $p(a) = d_n (a - \xi_1 e) \cdots (a - \xi_n e)$

Since $a - \xi_1 e, \dots, a - \xi_n e$ commute, Prop. 5(c)

yields $p(a) \in G(A) \Leftrightarrow \{a - \xi_1 e, \dots, a - \xi_n e\} \subset G(A)$

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 $\{\xi_1, \dots, \xi_n\} \subset \rho(a)$

$$(b) \sigma(p(a)) = p(\sigma(a))$$

$$\Gamma \mu \in \sigma(p(a)) \Leftrightarrow \mu e - p(a) \notin \zeta(A) \Leftrightarrow$$

(observe that $\mu e - p(a) = (\mu - p)(a)$, so)

$$\Leftrightarrow (\mu - p)(a) \notin \zeta(a) \stackrel{(a)}{\Leftrightarrow} \exists \lambda, \text{ a root of } \mu - p \\ \text{s.t. } \lambda \notin \zeta(a)$$

$$\Leftrightarrow \exists \lambda \in \sigma(a) : \mu - p(\lambda) = 0 \Leftrightarrow \mu \in p(\sigma(a)) \quad \lrcorner$$