

Let K, L be compact Hausdorff spaces

$\varphi: C(K) \rightarrow C(L)$ be a $*$ -homomorphism, $\varphi(1) = 1$

Then $\exists d: L \rightarrow K$ cts s.t. $\varphi(f) = f \circ d$, $f \in C(K)$

By Prop. 30 we know $\|\varphi\| \leq 1$, thus φ is cts.

Thus $\varphi': C(L)^* \rightarrow C(K)^*$ is well defined. (the dual operator)

We have $\varphi'(\Delta(C(L))) \subseteq \Delta(C(K))$

$$\begin{aligned} \forall h \in \Delta(C(L)) &\Rightarrow \varphi'(h)(1) = h(\varphi(1)) = h(1) = 1 \\ \varphi'(h)(fg) &= h(\varphi(fg)) = h(\varphi(f)\varphi(g)) = \\ &= h(\varphi(f))h(\varphi(g)) = \varphi'(h)(f)\varphi'(h)(g) \end{aligned}$$

For $x \in L$ let $\delta_x \in C(L)^*$ be the evaluation map
 $\delta_x(f) = f(x)$, $f \in C(L)$

Then $\delta_x \in \Delta(C(L))$, thus $\varphi'(\delta_x) \in \Delta(C(K))$.

Therefore $\exists! d(x) \in K$ s.t. $\varphi'(\delta_x) = \delta_{d(x)}$.

Thus we have a mapping $d: L \rightarrow K$

For $f \in C(K)$ and $x \in L$ we have

$$\varphi(f)(x) = \delta_x(\varphi(f)) = \varphi'(\delta_x)(f) = \delta_{d(x)}(f) = f(d(x))$$

So, $\varphi(f) = f \circ d$.

It remains to show that d is cts.

Observe: $x \mapsto \delta_x$ is cts $L \rightarrow (C(L)^*, w^*)$

[indeed, $\forall f \in C(L)$ $x \mapsto \delta_x(f) = f(x)$ is cts]

$x \mapsto \delta_x$ is one-to-one

$$[x \neq y \Rightarrow \exists f \in C(K) \quad f(x) \neq f(y)]$$

by Urysohn lemma.]

So, $x \mapsto \delta_x$ is a homeomorphism (as L is compact Hausdorff)

Further, φ' is \mathcal{U}^* -to- \mathcal{U}^* cts, as any dual operator

$$\Gamma \quad h \in C(K)^* \mapsto \varphi'(h) \text{ is } \mathcal{U}^* \text{-cts,}$$

as for each $f \in C(K)$

$$h \mapsto \varphi'(h)(f) = h(\varphi(f)) \text{ is } \mathcal{U}^* \text{-cts} \quad \downarrow$$

$$\text{Thus } d : x \mapsto \delta_x \mapsto \varphi'(\delta_x) = \delta_{\varphi(x)} \mapsto d(x)$$

is cts, as the composition of three cts maps.]

If φ is one-to-one, then $d(L) = K$, hence φ is an isometry

$$\Gamma \quad d(L) \subset K \text{ is closed} \quad (d \text{ cts, } K, L \text{ compact Hausdorff})$$

$$\Rightarrow \text{if } x \in K \setminus d(L) \Rightarrow \exists f \in C(K) \text{ s.t. } f(x) = 1, f|_{d(L)} = 0$$

(by Urysohn lemma)

Then $f \neq 0$ and $\varphi(f) = 0$, so φ is not one-to-one.

If $d(L) = K$, then

$$\|\varphi(f)\|_{C(L)} = \sup_{x \in L} |\varphi(f)(x)| = \sup_{x \in L} |f(\varphi(x))| =$$

$$= \sup_{y \in d(L)} |f(y)| = \sup_{y \in K} |f(y)| = \|f\|_{C(K)}. \quad]$$