IV.5 C*-algebras – basic properties

Definition. Let A be a Banach algebra.

- An **involution** on A is a mapping $x \mapsto x^*$ of A into itself such that for each $x, y \in A$ and $\lambda \in \mathbb{C}$ one has

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x^*, \qquad (xy)^* = y^* x^* \qquad \text{and} \qquad x^{**} = x.$$

- A Banach algebra A with involution is called a C^*-**algebra** if for each $x \in A$ one has

$$\|x^* x\| = \|x\|^2.$$

- If A is a Banach algebra with involution and $x \in A$, the element x is called **selfadjoint** (or **hermitien**) if $x^* = x$; x is called **normal** if $x^* x = x x^*$.

Remarks.

(1) Let A be a Banach algebra with involution. Then $e \in A$ is a left unit if and only if e^* is a right unit. Hence, if A has either a left unit or a right unit, it is unital and the unit is selfadjoint.

(2) If A is a Banach algebra with involution such that

$$\|x^* x\| \geq \|x\|^2 \text{ for } x \in A,$$

then A is a C^*-algebra.

(3) Let A be a C^*-algebra. Then $x \mapsto x^*$ is a conjugate linear isometry of A onto A. Hence,

$$\|x^* x\| = \|x x^*\| = \|x\|^2 = \|x^*\|^2 \text{ for } x \in A.$$

Examples 25.

(1) *The complex field is a commutative C^*-algebra, if the involution is defined by $\lambda^* = \overline{\lambda}$ for $\lambda \in \mathbb{C}$.*

(2) *The algebra $\mathcal{C}_0(T)$ (where T is locally compact space) is a commutative C^*-algebra, if the involution is defined by $f^*(t) = \overline{f(t)}$ for $t \in T$.*

(3) *The matrix algebra M_n is a C^*-algebra if the involution is defined by*

$$\left((a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \right)^* = (\overline{a_{ji}})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}.$$

(4) *If H is a Hilbert space, then the algebras $L(H)$ and $K(H)$ are C^*-algebras, if the involution T^* is defined to be the adjoint operator to T.*

(5) *On the algebra $L^1(\mathbb{R}^n)$ one can define an involution by $f^*(x) = \overline{f(x)}$, $x \in \mathbb{R}^n$; or by $f^*(x) = \overline{f(-x)}$, $x \in \mathbb{R}^n$. $L^1(\mathbb{R}^n)$ is not a C^*-algebra with any of these involutions.*

Proposition 26 (properties of algebras with involution). *Let A be a Banach algebra with involution and let $x \in A$. Then:*

(a) *Elements $x + x^*$, $i(x - x^*)$, $x^* x$ are selfadjoint.*

(b) *There exist uniquely determined selfadjoint elements $u, v \in A$ such that $x = u + iv$. Moreover, x is normal if and only if $uv = vu$.*

(c) *If A is unital, then $x \in G(A)$ if and only if $x^* \in G(A)$ (then $(x^*)^{-1} = (x^{-1})^*$).*

(d) $\sigma(x^*) = \{\overline{\lambda} : \lambda \in \sigma(x)\}$.

Proposition 27 (on the spectral radius and the norm of a normal element). *If A is a C^*-algebra and $a \in A$ is normal, then $r(a) = \|a\|$.*

Corollary 28. *Let A be an algebra with involution. Then there is at most one norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a C^*-algebra.*

Proposition 29 (adding a unit). *Let A be a Banach algebra with involution.*

(a) *A^+ is again a Banach algebra with involution, provided the involution is defined by $(a, \lambda)^* = (a^*, \overline{\lambda})$ for $(a, \lambda) \in A^+$.*

(b) *If A is a C^*-algebra, then A^+ is also a C^*-algebra, if the involution is defined as in (a) and the norm on A^+ is defined by*

$$\|(a, \lambda)\| = \max\{|\lambda|, \sup\{\|ab + \lambda b\|; b \in A, \|b\| \leq 1\}\}.$$

(c) *If A is a C^*-algebra with no unit, then the norm defined in (b) can be expressed as*

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\|; b \in A, \|b\| \leq 1\}.$$

Remark: The norm on A^+ defined in Proposition 29(b) differs from the norm given in Proposition 2(b). It follows from Corollary 28 that the formula from Proposition 29(b) is the unique possible.

Definition. Let A and B be C^*-algebras and let $h : B \to A$. We say that h is a **$*$-homomorphism**, if it is a homomorphism of Banach algebras satisfying moreover $h(x^*) = h(x)^*$ for each $x \in B$.

Proposition 30 (on the automatic continuity of a $*$-homomorphism). *Let A and B be C^*-algebras and let $h : B \to A$ be a $*$-homomorphism of B into A. Then $\|h\| \leq 1$.*

Example 31. *Let K, L be compact Hausdorff spaces and let $\varphi : \mathcal{C}(K) \to \mathcal{C}(L)$ be a $*$-homomorphism satisfying $\varphi(1) = 1$. Then there is a continuous mapping $\alpha : L \to K$ such that $\varphi(f) = f \circ \alpha$ for $f \in \mathcal{C}(K)$. If φ is moreover one-to-one, then $\alpha(L) = K$, so φ is an isometry of $\mathcal{C}(K)$ into $\mathcal{C}(L)$.*

Proposition 32. *Let A be a C^*-algebra and let $a \in A$. Then:*

(a) *If a is selfadjoint, then $\sigma(a) \subset \mathbb{R}$.*

(b) *If A is unital and $a^* = a^{-1}$ (i.e., a is* **unitary***), then $\sigma(a) \subset \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

(c) *For $h \in \Delta(A)$ and $a \in A$ one has $h(a^*) = \overline{h(a)}$ (i.e., h is a $*$-homomorphism whenever it is a homomorphism).*

Theorem 33 (Gelfand-Naimark). *Let A be a commutative C^*-algebra and let $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ be its Gelfand transform. Then Γ is an isometric $*$-isomorphism of the C^*-algebra A onto the C^*-algebra $\mathcal{C}_0(\Delta(A))$ (in particular $\widehat{x^*} = \overline{\hat{x}}$ for $x \in A$).*

Therefore, A is unital if and only if $\Delta(A)$ is compact.

Corollary 34. *Let A and B be commutative C^*-algebras. Then A and B are $*$-isomorphic if and only if $\Delta(A)$ and $\Delta(B)$ are homeomorphic.*

Corollary 35. *Let A and B be C^*-algebras and let $h : B \to A$ be a one-to-one $*$-homomorphism of B into A. Then h is an isometry of B into A.*