IV.6 Continuous functional calculus in C*-algebras

Proposition 36. *Let A be a C^*-algebra and $B \subset A$ its C^*-subalgebra.*

(a) *For each $x \in B$ one has $\sigma_B(x) \cup \{0\} = \sigma_A(x) \cup \{0\}$.*
(b) *If A has a unit e and, moreover, $e \in B$, then $\sigma_B(x) = \sigma_A(x)$ for each $x \in B$. In particular, $G(B) = B \cap G(A)$.*

Theorem 37 (Fuglede). *Let A be a C^*-algebra and let $x \in A$ be a normal element. If $y \in A$ commutes with x, it commutes also with x^*.*

Theorem 38 (continuous functional calculus in unital C^*-algebras). *Let A be a C^*-algebra with a unit e and let $x \in A$ be a normal element. Let B be the closed subalgebra of A generated by the set $\{e, x, x^*\}$. Then:*

- *B is a commutative C^*-algebra and e is its unit.*
- *The mapping $h : \varphi \mapsto \varphi(x)$ is a homeomorphism of $\Delta(B)$ onto $\sigma(x)$.*

Let $\Gamma : B \to \mathcal{C}(\Delta(B))$ be the Gelfand transform of the algebra B. For $f \in \mathcal{C}(\sigma(x))$ define

$$\tilde{f}(x) = \Gamma^{-1}(f \circ h).$$

Then the mapping $\Phi : f \mapsto \tilde{f}(x)$, called the **continuous functional calculus for** *x, enjoys the following properties:*

(a) *Φ is an isometric $*$-isomorphism of the C^*-algebra $\mathcal{C}(\sigma(x))$ onto B.*
(b) *$\tilde{id}(x) = x$, $\tilde{1}(x) = e$.*
(c) *If p is a polynomial, then $\tilde{p}(x) = p(x)$.*
(d) *$\sigma(\tilde{f}(x)) = f(\sigma(x))$ for $f \in \mathcal{C}(\sigma(x))$.*
(e) *If $y \in A$ commutes with x, then y commutes with $\tilde{f}(x)$ for each $f \in \mathcal{C}(\sigma(x))$.*

Moreover, Φ is the unique mapping satisfying the first two conditions.

Remark: By Proposition 36 $\sigma_A(x) = \sigma_B(x)$ in the preceding theorem, therefore we write just $\sigma(x)$.

Theorem 39 (continuous functional calculus in general C^*-algebras). *Let A be a C^*-algebra (unital or not) and let $x \in A$ be a normal element. Let B be the closed subalgebra of A generated by the set $\{x, x^*\}$. Then:*

- *B is a commutative C^* algebra.*
- *The mapping $h : \varphi \mapsto \varphi(x)$ is a homeomorphism of $\Delta(B) \cup \{0\}$ onto $\sigma(x) \cup \{0\}$.*

Let $\Gamma : B \to \mathcal{C}_0(\Delta(B))$ be the Gelfand transform of the algebra B. For $f \in \mathcal{C}_0(\sigma(x) \setminus \{0\})$ define

$$\tilde{f}(x) = \Gamma^{-1}(f \circ h).$$

Then the mapping $\Phi : f \mapsto \tilde{f}(x)$, called the **continuous functional calculus for** *x, enjoys the following properties:*

(a) *Φ is an isometric $*$-isomorphism of the C^*-algebra $\mathcal{C}_0(\sigma(x) \setminus \{0\})$ onto B.*
(b) *$\tilde{id}(x) = x$.*
(c) *If p is a polynomial satisfying $p(0) = 0$, then $\tilde{p}(x) = p(x)$.*
(d) *$\sigma(\tilde{f}(x)) \cup \{0\} = f(\sigma(x) \setminus \{0\}) \cup \{0\}$ for $f \in \mathcal{C}_0(\sigma(x) \setminus \{0\})$.*
(e) *If $y \in A$ commutes with x, then y commutes with $\tilde{f}(x)$ for each $f \in \mathcal{C}_0(\sigma(x) \setminus \{0\})$.*

Moreover, Φ is the unique mapping satisfying the first two conditions.

Remarks:

(1) By Proposition 36 $\sigma_A(x) \cup \{0\} = \sigma_B(x) \cup \{0\}$ in the preceding theorem, hence also $\sigma_A(x) \setminus \{0\} = \sigma_B(x) \setminus \{0\}$. Therefore we write just $\sigma(x)$.
(2) The algebra B from Theorem 39 is unital, if and only if $\sigma(x) \setminus \{0\}$ is compact. Its unit may differ from the unit of A (if it exists). There are the following possibilities:
  (a) $0 \notin \sigma_B(x) = \sigma_A(x)$. Then A is unital, the unit of A belongs to B and x is invertible (both in A and in B).
  (b) $0 \in \sigma_A(x) \setminus \sigma_B(x)$. Then B admits a unit which is not a unit of A (either A has no unit, or it has a unit which does not belong to B) and x is invertible in B (not in A).
(3) If $\sigma(x) \setminus \{0\}$ is compact, then $\mathcal{C}_0(\sigma(x) \setminus \{0\})$ is just $\mathcal{C}(\sigma(x) \setminus \{0\})$.
(4) If $0 \in \sigma_A(x)$ (this happens whenever $\sigma(x) \setminus \{0\}$ is not compact, but not only in this case), then one can identify $\mathcal{C}_0(\sigma(x) \setminus \{0\}) = \{f \in \mathcal{C}(\sigma_A(x)); f(0) = 0\}$.