I.4 Metrizability of topological vector spaces

Theorem 12 (characterization of metrizable TVS). *Let $(X, \mathcal{T})$ be a HTVS. The following assertions are equivalent:*

(i) *X is metrizable (i.e., the topology $\mathcal{T}$ is generated by a metric on X).*
(ii) *There exists a translation invariant metric on X generating the topology $\mathcal{T}$.*
(iii) *There exists a countable base of neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$.*

Proposition 13. *Let $(X, \mathcal{T})$ be a HTVS which has a countable base of neighborhoods of $\boldsymbol{o}$. Then there exists a function $p : X \to [0, \infty)$ with the following properties:*

(a) $p(\boldsymbol{o}) = 0$;
(b) $\forall x \in X \setminus \{\boldsymbol{o}\} : p(x) > 0$;
(c) $\forall x \in X \forall \lambda \in \mathbb{F}, |\lambda| \leq 1 : p(\lambda x) \leq p(x)$;
(d) $\forall x, y \in X : p(x + y) \leq p(x) + p(y)$;
(e) $\forall x \in X : \lim\limits_{t \to 0+} p(tx) = 0$;
(f) $\Big\{\{x \in X; p(x) < r\}; r > 0\Big\}$ *is a base of neighborhoods of $\boldsymbol{o}$ in X.*

Then the formula $\rho(x, y) = p(x - y)$, $x, y \in X$, defines a translation invariant metric on X generating the topology $\mathcal{T}$.

Remark. Given a vector space X, a function $p : X \to [0, \infty)$ satisfying conditions (a)–(e) from the previous proposition is called an **F-norm** on X. If p satisfies conditions (a),(c)–(e), it is called an **F-seminorm**.

Corollary 14. *Any HTVS which admits a bounded neighborhood of zero is metrizable.*

V.5 Minkowski functionals, seminorms and generating of locally convex topologies

Definition. Let X be a vector space and let $A \subset X$ be an absorbing set. By the **Minkowski functional** of the set A we mean the function defined by the formula

$$p_A(x) = \inf\{\lambda > 0; x \in \lambda A\}, \qquad x \in X.$$

Proposition 15 (basic properties of Minkowski functionals). *Let X be a vector space and let $A \subset X$ be an absorbing set.*

- *$p_A(tx) = tp_A(x)$ whever $x \in X$ and $t > 0$.*
- *If A is convex, p_A is a sublinear functional.*
- *If A is absolutely convex, p_A is a seminorm.*

Lemma 16. *Let X be a TVS and let $A \subset X$ be a convex set. If $x \in \overline{A}$ and $y \in \operatorname{Int} A$, then $\{tx + (1 - t)y; t \in [0, 1)\} \subset \operatorname{Int} A$.*

Proposition 17 (on the Minkowski functional of a convex neighborhood of zero). *Let X be a TVS amd let $A \subset X$ be a convex neighborhood of $\boldsymbol{o}$. Then:*

- *p_A is continuous on X.*
- $\operatorname{Int} A = \{x \in X; p_A(x) < 1\}$.
- $\overline{A} = \{x \in X; p_A(x) \leq 1\}$.
- $p_A = p_{\overline{A}} = p_{\operatorname{Int} A}$.

Corollary 18. *Any LCS is completely regular. Any HLCS is Tychonoff.*

Remark: It can be shown that even any TVS is completely regular, and so any HTVS is Tychonoff. The proof is more complicated, one can use a generalization of Proposition 13 from Section V.4. The proof that any TVS is regular is easy, it follows from Proposition 3(ii).

Theorem 19 (on the topology generated by a family of seminorms). *Let X be a vector space and let $\mathcal{P}$ be a nonempty family of seminorms on X. Then there exists a unique topology $\mathcal{T}$ na X such that $(X, \mathcal{T})$ is TVS and the family*

$$\Big\{\{x \in X; p_1(x) < c_1, \dots, p_k(x) < c_k\}; p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0\Big\}$$

is a base of neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$. The topology $\mathcal{T}$ is moreover locally convex. The topology $\mathcal{T}$ is Hausdorff if and only if for each $x \in X \setminus \{\boldsymbol{o}\}$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$.

Definition. The topology $\mathcal{T}$ from Theorem 19 is called **the topology generated by the family of seminorms** $\mathcal{P}$.

Theorem 20 (on generating of locally convex topologies). *Let $(X, \mathcal{T})$ be a LCS. Let $\mathcal{P}_{\mathcal{T}}$ be the family of all continuous seminorms on $(X, \mathcal{T})$. Then the topology generated by the family $\mathcal{P}_{\mathcal{T}}$ equals $\mathcal{T}$.*

Proposition 21. *Let X be a vector space.*

1. *If p is a seminorm on X, then the set $A = \{x \in X; p(x) < 1\}$ is absolutely convex, absorbing and satisfies $p = p_A$.*
2. *Let p, q be two seminorms on X. Then $p \le q$ if and only if*
$$\{x \in X; p(x) < 1\} \supset \{x \in X; q(x) < 1\}.$$
3. *Let $\mathcal{P}$ be a nonempty family of seminorms on X and let $\mathcal{T}$ be the topology generated by the family $\mathcal{P}$. Let p be a seminorm on X. Then p is $\mathcal{T}$-continuous if and only if there exist $p_1, \dots, p_k \in \mathcal{P}$ and $c > 0$ such that $p \le c \cdot \max\{p_1, \dots, p_k\}$.*

Theorem 22 (on metrizability of LCS). *Let $(X, \mathcal{T})$ be a HLCS. The following assertions are equivalent:*

(i) *X is metrizable (i.e., the topology $\mathcal{T}$ is generated by a metric on X).*
(ii) *There exists a translation invariant metric on X generating the topology $\mathcal{T}$.*
(iii) *There exists a countable base of neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$.*
(iv) *The topology $\mathcal{T}$ is generated by a countable family of seminorms.*

Proposition 23. *(1) Let $(X, \mathcal{T})$ be a metrizable LCS. Then the topology $\mathcal{T}$ is generated by a sequence of seminorms (p_n) satisfying*
$$p_1 \le p_2 \le p_3 \le \dots$$

(2) Let X be a vector space and let (p_n) be a sequence of seminorms on X satisfying conditions:

- $p_1 \le p_2 \le p_3 \le$;
- $\forall x \in X \setminus \{\boldsymbol{o}\}\, \exists n \colon p_n(x) > 0$.

Then
$$\rho(x, y) = \sum_{n=1}^{\infty} \tfrac{1}{2^n} \min\{1, p_n(x - y)\}, \quad x, y \in X$$
is a translation invariant metric on X which generates the locally convex topology on X generated by the sequence of seminorms (p_n). Moreover, given a sequence (x_k) in X we have

(a) $\rho(x_k, x) \to 0 \Leftrightarrow \forall n \in \mathbb{N} \colon p_n(x_k - x) \xrightarrow{k} 0$;
(b) *the sequence (x_k) is Cauchy in ρ if and only of it is Cauchy in each of the seminorms p_n.*

Theorem 24 (a characterization of normable TVS). *Let $(X, \mathcal{T})$ be a HTVS. Then X is normable (i.e., $\mathcal{T}$ is generated by a norm) if and only if X admits a bounded convex neighborhood of $\boldsymbol{o}$.*

Proposition 25. *Let X be a LCS.*

(a) *The set $A \subset X$ is bounded if and only if each continuous seminorm p on X is bounded on A. (It is enough to test it for a family of seminorms generating the topology of X.)*
(b) *Let Y be a LCS and let $L : X \to Y$ be a linear mapping. Then L is continuous if and only if*
$$\forall q \text{ a continuous seminorm on } Y\, \exists p \text{ a continuous seminorm on } X\, \forall x \in X : q(L(x)) \le p(x).$$
If $\mathcal{P}$ is a family of seminorms generating the topology of X and $\mathcal{Q}$ is a family of seminorms generating the topology of Y, then the continuity of L is equivalent to the condition
$$\forall q \in \mathcal{Q}\, \exists p_1, \dots, p_k \in \mathcal{P}\, \exists c > 0\, \forall x \in X : q(L(x)) \le c \cdot \max\{p_1(x), \dots, p_k(x)\}.$$
(c) *A net (x_τ) converges to $x \in X$ if and only if $p(x_\tau - x) \to 0$ for each continuous seminorm p on X. (It is enough to test it for a family of seminorms generating the topology of X.)*