FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2021/2022

PROBLEMS TO CHAPTER I

Problems to Section I.1 – linear topologies and their generation

Problem 1. Let X be a vector space and let $A \subset X$ be a nonempty set. Show that A is convex if and only if $(\alpha+\beta)A = \alpha A + \beta A$ for each $\alpha, \beta > 0$.

Problem 2. Let X be a vector space and let $\mathcal{T}$ be a topology on X.

(1) Show that $\mathcal{T}$ is translation invariant if and only if the addition on X is separately continuous (i.e. $x \mapsto x + y$ is continuous for each $y \in X$).
(2) Find X and $\mathcal{T}$ such that the addition is continuous (in both variables simultaneously, as a mapping $X \times X \to X$) but the multiplication is not continuous.
(3) Let $\mathcal{T}$ be translation invariant. Show that the addition is continuous (in both variables) if and only if for any neighborhood U of zero there is a neighborhood V of zero such that $V + V \subset U$.
(4) Find a topology on $\mathbb{R}^2$, in which the addition is separately continuous but not continuous.
(5) Let $\mathcal{T}$ be translation invariant and let there exists a base of neighborhoods of zero consisting of convex sets. Show that the the mapping $x \mapsto \frac{x}{2}$ is continuous.
(6) Let $\mathcal{T}$ be translation invariant and let there exists a base of neighborhoods of zero consisting of convex sets. Show that the addition is continuous (in both variables) if and only if the mapping $x \mapsto 2x$ is continuous.
(7) Let $\mathcal{T}$ be translation invariant and let there exists a base of neighborhoods of zero consisting of convex sets. Is the mapping $x \mapsto 2x$ necessarily continuous?
(8) Assume that the addition is continous. Show that the mapping $x \mapsto nx$ is continuous for each $n \in N$.

`Hint:` *(2) Consider the discrete topology. (4) Consider, for example, a neighborhood base of zero made by sets* $\{(0,0)\} \cup \{(x,y); |y| < |x| < r\}$, $r > 0$.

Problem 3. Let X be a vector space and let ρ be a translation invariant metric on X.

(1) Show that the addition is continuous (in both variables) in ρ.
(2) Using a counterexample show that the topology induced by ρ need not be linear.
(3) Show that ρ generates a linear topology on X if and only if

$$\lim_{x\to \boldsymbol{o}} \sup_{\lambda\in\mathbb{F},|\lambda|\le 1} \rho(\lambda x, \boldsymbol{o}) = 0 \text{ and } \forall x \in X\colon \lim_{\lambda\to 0} \rho(\lambda x, \boldsymbol{o}) = 0.$$

`Hint:` *(1) Either use (3) from the previous problem or the sequential characterization of continuity. In both cases use moreover the triangle inequality.*

Problem 4. Let X be a vector space. Let $\mathcal{U}$ be the family of all the absolutely convex absorbing subsets of X.

(1) Show that $\mathcal{U}$ is a base of neighborhoods of zero in a Hausdorff locally convex topology $\mathcal{T}$ on X.
(2) Show that this topology $\mathcal{T}$ is the strongest locally convex topology on X.
(3) Show that any convergent sequence in $(X, \mathcal{T})$ is contained in a finite-dimensional subspace.

`Hint:` *(3) Suppose it is not the case. Then there exists a linearly independent sequence (x_n) which converges to zero. Complete this sequence to an algebraic basis of X. Describe an absolutely convex absorbing set, not containing any of the vectors x_n.*

Problem 5. (1) Let $A \subset \mathbb{R}^2$ be balanced and absorbing. Show that $A + A$ is a neighborhood of zero (in the standard topology).
(2) Find a balanced absorbing $A \subset \mathbb{R}^2$, which is not a neighborhood of zero.
(3) Using the previous two points deduce that the family of all the balanced absorbing subsets of $\mathbb{R}^2$ is not the base of neighborhoods of zero in any linear topology on $\mathbb{R}^2$.

`Hint:` *(1) Show that $A + A$ contains a rectangle of the form $[-a, a] \times [-b, b]$.*

Problem 6. (1) Show that the convex hull of a balanced subset of a vector space is again balanced, and hence absolutely convex.
(2) Show that the balanced hull of a convex set need not be convex.

`Hint:` *(2) Consider a suitable segment in $\mathbb{R}^2$.*

Problem 7. Let X be the space of all the Lebesgue measurable functions on $[0, 1]$ (with values in $\mathbb{F}$; we identify the functions, which are equal almost everywhere). For $f, g \in X$ set

$$\rho(f, g) = \int_0^1 \min\{1, |f - g|\}.$$

(1) Show that ρ is a metric generating a linear topology on X.
(2) Show that the convergence of sequences in the metric ρ coincide with the convergence in measure.
(3) Is the resulting topology locally convex?

`Hint:` *(1) Show that the operations are continuous using the convergence of sequences. (3) Show that for each $r > 0$ the convex hull of the set $\{f \in X; \rho(f, 0) < r\}$ is the whole X.*

Problem 8. Let X be a TVS and let $A \subset X$ be a balanced set with nonempty interior.

(1) Show that $\operatorname{int} A$ is balanced if and only if $0 \in \operatorname{int} A$.
(2) Show on a counterexample that $\operatorname{int} A$ need not be balanced.

Problem 9. Let $(X, \mathcal{T})$ be a TVS and let $A \subset X$ be nonempty. Show that

$$\overline{A} = \bigcap\{A + U; U \in \mathcal{T}(0)\}.$$

Problem 10. Let $(X, \mathcal{T})$ be a non-Hausdorff TVS.

- Denote $Z = \overline{\{\boldsymbol{o}\}} = \bigcap \mathcal{T}(0)$. Show that Z is a vector subspace of X.
- Let $Y = X/Z$ be the quotient vector space and let $q : X \to Y$ be the canonical quotient mapping. Let $\mathcal{R}$ be the quotient topology on Y (i.e., $\mathcal{R} = \{U \subset Y; q^{-1}(U) \in \mathcal{T}\}$). Show that $(Y, \mathcal{R})$ is a HTVS.
- Show that $(Y, \mathcal{R})$ is locally convex if and only if $(X, \mathcal{T})$ is locally convex.

Problem 11. Let X be a TVS and let $A \subset X$. Show that A is bounded if and only if each countable subset of A is bounded.

Problem 12. Let X be a TVS and let $A, B \subset X$ be bounded sets. Show that the sets $A \cup B$, $A + B$, $\overline{A}$, $\mathrm{b}(A)$ are bounded as well.

Problem 13. Let X be a LCS and let $A \subset X$ be a bounded set. Show that the sets $\operatorname{co} A$ and $\operatorname{aco} A$ are bounded as well.

Problem 14. Let $X = L^p([0,1])$ where $p \in (0,1)$. Show that $A = \{f \in X; \|f\|_p < 1\}$ is a bounded set whose convex hull is not bounded.

`Hint:` *Show that* $\operatorname{co} A = X$.

Problem 15. Let X be a normed linear space and $A \subset X$. Show that A is bounded as a subspace of the TVS X if and only if it is bounded in the metric generated by the norm.

Problem 16. Let X be a TVS whose topology is generated by a translation invariant metric ρ.

(1) Show that any set $A \subset X$ bounded in X is bounded in the metric ρ as well.
(2) Show that a set $A \subset X$ bounded in the metric ρ need not be bounded in the TVS X.

`Hint:` *(2) The metric ρ itself may be bounded.*

Problem 17. Consider the space of test functions $\mathscr{D}(\Omega)$ with the topology from Example I.5(2). Let $\Lambda : \mathscr{D}(\Omega) \to \mathbb{F}$ be a linear functional. Show that $\Lambda \in \mathscr{D}'(\Omega)$ (i.e., Λ is a distribution) if and only if Λ is continuous.

Problem 18. Consider the space $(X, \mathcal{T})$ from Problem 4. Show that any linear functional $L : X \to \mathbb{F}$ is continuous.

Problem 19. Let $X = \mathbb{F}^\Gamma$ and let $A \subset X$. Show that A is bounded in X if and only if it is „pointwise bounded", i.e., if and only if the set $\{x(\gamma); x \in A\}$ is bounded in $\mathbb{F}$ for each $\gamma \in \Gamma$.

Problem 20. Let $X = \mathcal{C}([0,1])$ equipped with the topology of pointwise convergence, let $Y = \mathcal{C}([0,1])$ equipped with the topology generated by the metric ρ from Problem 7 and let $L : X \to Y$ be the identity mapping.

(1) Show that L maps bounded sets to bounded sets.
(2) Show that L is sequentially continuous (i.e., $f_n \to f$ in X implies $Lf_n \to Lf$ in Y).
(3) Show that L is not continuous.

`Hint:` *(1) Let $A \subset X$ be bounded. For $n \in \mathbb{N}$ set $F_n = \{x \in [0,1]; \forall f \in A : |f(x)| \le n\}$. Show that F_n are closed subsets of $[0,1]$, whose union is the whole interval $[0,1]$. For $\varepsilon > 0$ choose $n \in \mathbb{N}$ such that $\lambda([0,1] \setminus F_n) < \frac{\varepsilon}{2}$; and $m > n$ such that $\frac{n}{m} < \frac{\varepsilon}{2}$. Show that then $L(A) \subset m\{g \in Y; \rho(g,0) < \varepsilon\}$. Deduce the boundedness of $L(A)$ in Y. (2) Use Lebesgue dominated convergence theorem. (3) Let $U = \{f \in Y; \rho(f,0) < \frac{1}{2}\}$. Then U is a neighborhood of zero in Y. Show that $L^{-1}(U)$ is not a neighborhood of zero in X. (For any finite $F \subset [0,1]$ find $f \in \mathcal{C}([0,1])$ such that $f|_F = 0$ but $f = 1$ on a set of measure at least $\frac{1}{2}$.)*

Problem 21. Let X be a TVS and let (x_n) be a sequence of elements of X. Show that the sequence (x_n) is bounded in X if and only if for any sequence (λ_n) in $\mathbb{F}$ one has $\lambda_n \to 0 \Rightarrow \lambda_n x_n \to \boldsymbol{o}$.

Problem 22. Let X be a metrizable TVS and let (x_n) be a sequence of elements of X. Show that there exists a sequence of strictly positive numbers (λ_n) such that $\lambda_n x_n \to \boldsymbol{o}$.

`Hint:` *Let ρ be a metric generating the topology on X. Show and then use that $\lim_{t\to 0+} \rho(\boldsymbol{o}, tx) = 0$ for each $x \in X$.*

Problem 23. Is the assertion from Problem 22 true also for non-metrizable TVS?

`Hint:` *Consider the space from Problem 4.*

Problem 24. Let X be a TVS, whose topology is generated by a translation invariant metric ρ. Let (x_n) be a sequence of elements X converging to zero. Show that there exists a sequence of positive numbers (λ_n) such that $\lambda_n \to \infty$ and $\lambda_n x_n \to \boldsymbol{o}$.

`Hint:` *By the translation invariance of ρ it follows $\rho(\boldsymbol{o}, nx) \le n\rho(\boldsymbol{o}, x)$ for $x \in X$ and $n \in \mathbb{N}$.*

Problem 25. Is the assertion from the previous problem valid for a general TVS?

`Hint:` *Consider, e.g., $X = c_0$ or $X = \ell^p$ for $p \in (1, \infty)$ with the weak topology (see Section II.1), let (x_n) be the sequence of canonical unit vectors.*

Problems to Section I.3 – finite-dimensional and infinite-dimensional spaces

Problem 26. Show that any finite-dimensional TVS is locally convex.

`Hint:` *For a Hausdorff space use Proposition I.9. For the general case use Problem 10.*

Problem 27. Let X be a vector space of infinite dimension. Show that on X there exists a Hausdorff linear topology, which is not locally convex.

`Hint:` *Let $(e_j)_{j\in J}$ be an algebraic basis of X. Choose $p \in (0, 1)$ and define a metric on X by $\rho(\sum a_j e_j, \sum b_j e_j) = \sum |a_j - b_j|^p$.*

Problem 28. Let X be a metrizable TVS of infinite dimension. Show that there exists a discontinuous linear functional on X.

`Hint:` *Use an algebraic basis of X and Problem 22.*

Problem 29. Is there a a discontinuous linear functional on each infinite-dimensional HTVS (or HLCS)?

`Hint:` *Use Problem 18.*

Problems to Section I.4 – metrizability of TVS

Problem 30. Let X be a vector space over $\mathbb{F}$ and let $p : X \to [0, \infty)$ be an F-norm on X.

(1) Show that the formula $\rho(x, y) = p(x - y)$ defines a translation invariant metric on X, which generates a linear topology na X.
(2) Show that the family $\left\{\{x \in X; p(x) < \frac{1}{n}\}; n \in \mathbb{N}\right\}$ is a base of neighborhoods of zero in this topology.

Problem 31. Let X be a vector space over $\mathbb{F}$ and let $q : X \to [0, \infty)$ be a **quasinorm** on X, i.e., a mapping with the following properties:

- $\forall x \in X : q(x) = 0 \Leftrightarrow x = \boldsymbol{o}$;
- $\forall x \in X\, \forall \lambda \in \mathbb{F} : q(\lambda x) = |\lambda|\, q(x)$;
- $\exists C \geq 1\, \forall x, y \in X : q(x + y) \leq C(q(x) + q(y))$.

(1) Show that the family $\Big\{\{x \in X; q(x) < r\}; r \in (0, \infty)\Big\}$ is a base of neighborhoods of zero in a linear topology on X.
(2) Show that this topology is metrizable.
(3) Show that $x_n \to x$ in this topology if and only if $q(x_n - x) \to 0$.

`Hint:` *(2) Show that there exists a countable base of neighborhoods of zero.*

Problem 32. Let X be a vector space over $\mathbb{F}$, let $q : X \to [0, \infty)$ be an F-norm on X and let $p \in (0, 1)$. We say thate q is a p-**norm**, if $q(\lambda x) = |\lambda|^p\, q(x)$ for each $x \in X$ and $\lambda \in \mathbb{F}$.

(1) Show that the function $f \mapsto \|f\|_p$ is a p-norm on $L^p(\mu)$.
(2) Show that the function $f \mapsto (\|f\|_p)^{1/p}$ is a quasinorm on $L^p(\mu)$.
(3) Let q be a p-norm on X. Show that $q^{1/p}$ is a quasinorm on X and estimate C.

`Hint:` *(2) is a special case of (3). Set $\alpha = \frac{1}{p}$. By analysing monotonicity of a suitable function show that $a^\alpha + b^\alpha \leq (a + b)^\alpha \leq 2^{\alpha - 1}(a^\alpha + b^\alpha)$ whenever $a, b \geq 0$.*

Problem 33. Let X be a TVS and let p be an F-seminorm on X. Show that p is continuous if and only if it is continous at zero.

Problem 34. Let X be a TVS and let U be a neigborhood of zero. Show that there exists a continuous F-seminorm p on X such that $\{x \in X; p(x) < 1\} \subset U$.

`Hint:` *Let (V_n) be sequence of balanced neighborhoods of zero satisfying $V_1 + V_1 \subset U$ and $V_{n+1} + V_{n+1} \subset V_n$ for $n \in N$. Apply the construction from the proof of Proposition I.14 to this sequence.*

Problem 35. Using the previous problem show that any TVS is completely regular.

Problems to Section I.5 – seminorms, Minkowski functionals, F-seminorms

Problem 36. On a counterexample show that the Minkowski functional of a balanced neighborhood of zero need not be continuous.

`Hint:` *It may happen that there is $x \in X$ and numbers $0 < a < b$ such that the segment $\{tx; t \in [a, b]\}$ is contained in the boundary of a given balanced neighborhood of zero. A counterexample may be constructed already in $\mathbb{R}^2$.*

Problem 37. Let X be a TVS, let $A \subset X$ be a balanced neighborhood of zero and p_A its Minkowski functional. Show that the following three conditions are equivalent:

(i) p_A is continuous on X;
(ii) For each $x \in \overline{A}$ one has $\{tx; t \in [0, 1)\} \subset \operatorname{int} A$;
(iii) $\operatorname{int} A = \{x \in X; p_A(x) < 1\}$ & $\overline{A} = \{x \in X; p_A(x) \leq 1\}$.

Problem 38. Let X be a TVS, let $A \subset X$ be a subset containing the origin and let $p \in (0,1)$. We say that the set A is p-convex if

$$\forall x, y \in A \, \forall s, t \in [0,1] : s^p + t^p = 1 \Rightarrow sx + ty \in A.$$

(1) Let A be p-convex, let $x_1, \dots, x_n \in A$ and let $t_1, \dots, t_n \in [0,1]$ satisfy $t_1^p + \cdots + t_n^p = 1$. Show that $t_1 x_1 + \cdots + t_n x_n \in A$.
(2) Let $0 < p < q < 1$. Show that

$$A \text{ convex} \Rightarrow A \text{ } q\text{-convex} \Rightarrow A \text{ } p\text{-convex}.$$

(3) On an example demonstrate that a p-convex set need not be convex.
(4) Let A be a p-convex neighborhood of zero. Show that its Minkowski functional is continuous.

Hint: *(1) Use the mathematical induction. (2) Use that $0 \in A$. (3) Consider for example the set $\{f \in L^p([0,1]); \|f\|_p < 1\}$. (4) Use the characterization from the previous problem.*

Problem 39. Show that the topology from Problem 4 is generated by the family of all the seminorms on X.

Problem 40. Let X be a vector space and let $\mathcal{P}$ be a family of F-seminorms na X.

(1) Show that the family

$$\Big\{\{x \in X; p_1(x) < c_1, \dots, p_k(x) < c_k\}; p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0\Big\}$$

forms a base of neighborhoods of zero in a linear topology on X.
(2) Show that the F-seminorms from $\mathcal{P}$ are continuous in this topology.
(3) Show that this topology is Hausdorff if and only if for each $x \in X \setminus \{\boldsymbol{o}\}$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$.

Problem 41. Let $(X, \mathcal{T})$ be a TVS. Let $\mathcal{P}$ be the family of all the continuous F-seminorms on X. Show that the topology generated by the family $\mathcal{P}$ in the sense of Problem 40 equals $\mathcal{T}$.

Hint: *Use Problem 34.*

Problem 42. Let X be a vector space. Show that on X there exists the strongest linear topology and that this topology is Hausdorff.

Hint: *Apply the construction from Problem 40 to the family of all the F-seminorms on X.*

Problem 43. Let X be a vector space of countable algebraic dimension. Show that the strongest linear topology on X is locally convex.

Hint: *Let $\mathcal{T}$ be the topology from Problem 4. Show that each F-seminorm on X is continuous in $\mathcal{T}$: Let p be an F-seminorm on X and let (e_n) be an algebraic basis. For each $\varepsilon > 0$ show that there is (t_n), a sequence of strictly positive numbers such that* $\operatorname{aco}\{t_n e_n; n \in \mathbb{N}\} \subset \{x; p(x) < \varepsilon\}$.

Problem 44. Let X be vector space of uncountable algebraic dimension. Show that the strongest linear topology on X is not locally convex.

Hint: *Show that the F-seminorm defined as in the Problem 27 is not continuous in the topology from Problem 4.*

Problems to Section I.6 - F-spaces, Fréchet spaces, totally bounded sets

Problem 45. Let $(X, \|\cdot\|)$ be a Banach space. Let $|||\cdot||| : X \to [0, +\infty]$ be a mapping satisfying the following properties:

- $|||\boldsymbol{o}||| = 0$;
- $\forall x \in X\, \forall \alpha \in \mathbb{F} \setminus \{0\} \colon |||\alpha x||| = |\alpha| \cdot |||x|||$;
- $\forall x, y \in X \colon |||x + y||| \le |||x||| + |||y|||$;
- $\forall x \in X : \|x\| \le |||x|||$;
- the function $|||\cdot|||$ is lower semicontinuous, i.e. the set $\{x \in X; |||x||| \le c\}$ is closed for each $c \in \mathbb{R}$.

Set $Y = \{x \in X; |||x||| < +\infty\}$.

(1) Show that Y is a linear subspace of X.
(2) Show that $|||\cdot|||$ is a norm on Y and $(Y, |||\cdot|||)$ is a Banach space.

`Hint:` *(2) To prove the completeness: Any $|||\cdot|||$-Cauchy sequence (x_n) is $\|\cdot\|$-Cauchy as well, hence it converges in X to some $x \in X$. It is necessary to show that $x \in Y$ and $|||x_n - x||| \to 0$. To this end use the definition of a limit and lower semicontinuity, i.e.: $x_n \to x$ in X and $|||x_n||| \le c$ for each $n \Rightarrow |||x||| \le c$.*

Problem 46. Prove the completeness of $\ell^p(\Gamma)$ for $1 \le p < +\infty$ using completeness of $\ell^\infty(\Gamma)$ and Problem 45.

Problem 47. Let $(X, \|\cdot\|)$ be a Banach space. Let $(|||\cdot|||_k)$ be a sequence of functions on X satisfying conditions from Problem 45 and moreover

$$\forall x \in X \colon \; \|x\| \le |||x|||_1 \le |||x|||_2 \le |||x|||_3 \le \dots$$

Set $Y = \{x \in X; \forall n \in \mathbb{N} \colon |||x||| < +\infty\}$.

(1) Show that Y is a linear subspace of X.
(2) Show that Y is a Fréchet space if it is equipped with the locally convex topology generated by the sequence of norms $(|||\cdot|||_n)$.

`Hint:` *(2) Use Proposition I.23 and the approach from Problem 45 to each of the norms $|||\cdot|||_n$.*

Problem 48. Let X be a TVS and let $A, B \subset X$ be totally bounded subsets. Show that the sets $A \cup B$, $A + B$, $\overline{A}$, $\mathrm{b}(A)$ are totally bounded as well.

Problem 49. Let X be a TVS, whose topology is generated by a translation invariant metric ρ. Show that a set $A \subset X$ is totally bounded in the TVS X if and only if it is totally bounded in the metric ρ.

Problem 50. On a counterexample show that in an F-space which is not locally convex, the closed convex hull of a compact set need not be compact.

`Hint:` *A counterexample may be constructed for example in $L^p([0,1])$ for $p \in (0,1)$: Choose strictly positive numbers ε, η and δ such that $p < \frac{1}{1+\varepsilon}$, $\frac{\eta}{\varepsilon} < p$, $\delta < \varepsilon$ and $\frac{\eta}{\varepsilon - \delta} < p$. For $n \in \mathbb{N}$ set $x_n = \frac{1}{n^{1+\eta}}$, $f_n = n^{1+\varepsilon}\chi_{(x_{n+1}, x_n)}$ and $t_n = \frac{1}{n^{1+\delta}}$. Show that $f_n \to 0$ in $L^p([0,1])$, and so $\{0, f_1, f_2, f_3, \dots\}$ is a compact set. Further show, using the elements $\frac{t_1 f_1 + \dots + t_n f_n}{t_1 + \dots + t_n}$, that the convex hull of the mentioned compact set is an ubbounded set in $L^p([0,1])$.*

Problems to Section I.7 – separation theorems

Problem 51. Let $X = L^p([0,1])$ where $p \in (0,1)$. Show that $X^* = \{0\}$.

`Hint:` *Show that the only two convex open sets in X are $\emptyset$ and X.*

Problem 52. Let $X = \ell^p$ where $p \in (0,1)$. Show that for each sequence $\boldsymbol{x} = (x_n) \in \ell^\infty$ the formula

$$\varphi_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{n=1}^{\infty} x_n y_n, \quad \boldsymbol{y} = (y_n) \in \ell^p,$$

defines a continuous linear fuctional on ℓ^p. Show that the mapping $\boldsymbol{x} \mapsto \varphi_{\boldsymbol{x}}$ is a linear bijection of ℓ^∞ onto X^*.

Problem 53. Let $p \in (0,1)$. Show that ℓ^p is isomorphic (even linearly isometric) to a subspace of $L^p([0,1])$. Using the two previous problems demonstrate on a counterexample that a continous linear functional on a subspace of a TVS need not admit a continous linear extension to the whole space.

Problem 54. Let X be a normed linear space of infinite dimension. Show that in X there exist two disjoint convex sets which are dense in X (and hence they cannot be separated by a nonzero element of X^*).

`Hint:` *Use the existence of a discontinuous linear functional.*

Problem 55. Let $X = \mathcal{C}([0,1])$ be equipped with the L^2-norm (i.e., $\|f\| = \left(\int_0^1 |f|^2\right)^{1/2}$). For $\alpha \in \mathbb{R}$ define $Y_\alpha = \{f \in X; f(0) = \alpha\}$. Show that $(Y_\alpha; \alpha \in \mathbb{R})$ is a pairwise disjoint family if dense convex sets. Show that for $\alpha \neq \beta$ the sets Y_α and Y_β cannot be separated by a nonzero element of X^*.

Problem 56. Let $X = c_0$ or $X = \ell^p$ for some $p \in [1,\infty)$ (consider the real version of these spaces). Let $\boldsymbol{x} = (x_n) \in X$ be an element with all the coordinates strictly positive and let $\boldsymbol{y} = (\frac{x_n}{n}) \in X$. Set

$$A = \{\boldsymbol{z} = (z_n) \in X; \forall n \in \mathbb{N} : z_n \geq 0\}, \qquad B = \{-\boldsymbol{x} + t\boldsymbol{y}; t \in \mathbb{R}\}.$$

Show that A and B are disjoint closed subsets of X, which cannot be separated by a nonzero element of X^*.

`Hint:` *Proceed by contradiction: Let $f \in X^* \setminus \{0\}$ satisfy $\sup f(B) \leq \inf f(A)$. Show that necessarily $f \geq 0$ on A and $\inf f(A) = 0$. The functional f can be represented by an appropriate sequence (by an element of ℓ^1 or ℓ^q where $\frac{1}{p} + \frac{1}{q} = 1$), show that all the entries of this sequence have to be non-negative. By the assumption $\inf f(B) \leq 0$ deduce $f(\boldsymbol{y}) = 0$, hence $f = 0$, a contradiction.*