

# FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2021/2022

PROBLEMS TO CHAPTER IV

## PROBLEMS TO SECTION IV.1 – EXAMPLES OF BANACH ALGEBRAS, INVERTIBLE ELEMENTS

**Problem 1.** Let  $A = (\mathbb{C}^n, \|\cdot\|_p)$ , where  $p \in [1, \infty]$  and  $n \geq 2$ . Equip  $A$  with the coordinatewise multiplication, i.e.,

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = (x_1y_1, x_2y_2, \dots, x_ny_n).$$

- (1) Show that  $A$  is a unital Banach algebra and find its unit.
- (2) Show that the unit has norm one if and only if  $p = \infty$ .
- (3) Apply on  $A$  the respective renorming and show, that the new norm is just  $\|\cdot\|_\infty$ .

**Problem 2.** Let  $M_n$  be the algebra of complex  $n \times n$ -matrices equipped with the matrix multiplication. Recall that any  $n \times n$ -matrix represents a linear mapping  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  and that the matrix multiplication corresponds to composition of linear mappings.

- (1) Fix  $p \in [1, \infty]$  and equip  $M_n$  with the operator norm coming from  $L((\mathbb{C}^n, \|\cdot\|_p))$ . Show that  $M_n$  is then a unital Banach algebra and that the unit has norm one.
- (2) Show that for  $p_1 \neq p_2$  the two norms defined in (1) are equivalent but different whenever  $n \geq 2$ .
- (3) Show that  $M_n$  is commutative if and only if  $n = 1$ .

**Problem 3.** Let  $M_n$  be the algebra of complex  $n \times n$ -matrices equipped with the matrix multiplication. Equip  $M_n$  with the norm

$$\|(a_{ij})_{i,j=1,\dots,n}\| = \sum_{i,j=1}^n |a_{ij}|.$$

Show that  $M_n$  equipped with this norm is a unital Banach algebra and its unit has norm greater than 1 (whenever  $n \geq 2$ ).

**Problem 4.** Let  $A = (\mathbb{C}^n, \|\cdot\|_\infty)$ , where  $n \geq 2$ .

- (1) Define multiplication on  $A$  by

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = (x_1y_1, x_1y_2, \dots, x_1y_n).$$

Show that  $A$  equipped with this multiplication is a Banach algebra and that  $A$  has many left units but no right unit.

- (2) Define multiplication on  $A$  by

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = (x_1y_1, x_2y_1, \dots, x_ny_1).$$

Show that  $A$  equipped with this multiplication is a Banach algebra and that  $A$  has many right units but no left unit.

- (3) Represent the algebras from (1) and (2) as subalgebras of the matrix algebra  $M_n$ .

**Hint:** (3) Consider matrices with only one nonzero row or column.

**Problem 5.** Let  $A = \ell^p(\Gamma)$ , where  $p \in [1, \infty]$  and  $\Gamma$  is an infinite set. Equip  $A$  with the pointwise multiplication.

- (1) Show that  $A$  is a Banach algebra.
- (2) Show that  $A$  is unital if and only if  $p = \infty$ .

**Problem 6.** Let  $X$  be any nontrivial Banach space. Define on  $X$  the trivial multiplication, i.e.,  $x \cdot y = \mathbf{o}$  for  $x, y \in X$ .

- (1) Show that  $X$  is a Banach algebra with no unit.
- (2) Describe the unital algebra  $X^+$ .
- (3) Represent the algebras  $X$  and  $X^+$  as subalgebras of  $L(X^+) = L(X \oplus_1 \mathbb{C})$ .
- (4) Find a subalgebra of the matrix algebra  $M_n$  (where  $n \geq 2$ ) isomorphic with such a trivial algebra.

*Hint:* (4) Use the description from (3) for  $X = \mathbb{C}^{n-1}$ .

**Problem 7.** Let  $A_1, \dots, A_n$  be Banach algebras and let  $p \in [1, \infty]$ . Consider the vector space  $A = A_1 \times A_2 \times \dots \times A_n$ , where the norm and multiplication are defined by

$$\begin{aligned} \|(a_1, \dots, a_n)\| &= \|(\|a_1\|, \dots, \|a_n\|)\|_p, \\ (a_1, \dots, a_n) \cdot (b_1, \dots, b_n) &= (a_1 b_1, \dots, a_n b_n). \end{aligned}$$

- (1) Show that  $A$  is a Banach algebra.
- (2) Show that  $A$  is unital if and only if  $A_1, \dots, A_n$  are unital. (Find the respective unit.)
- (3) Show that  $A$  is commutative if and only if  $A_1, \dots, A_n$  are commutative.

**Problem 8.** Let  $T$  be a Hausdorff locally compact space. Consider the Banach space  $\mathcal{C}_0(T)$  (with the max-norm) and equip it with the pointwise multiplication.

- (1) Show that  $\mathcal{C}_0(T)$  is a commutative Banach algebra.
- (2) Show that the algebra  $\mathcal{C}_0(T)$  is unital if and only if  $T$  is compact.
- (3) Assume that  $T$  is not compact. Let  $B = \text{span}(\mathcal{C}_0(T) \cup \{1\})$  as a subalgebra of  $\ell^\infty(T)$ . Show that  $B$  is (algebraically) isomorphic to  $(\mathcal{C}_0(T))^+$  but not isometric.

**Problem 9.** Let  $K$  be a compact Hausdorff space and let  $A$  be a Banach algebra. Let  $\mathcal{C}(K, A)$  be the vector space of all the continuous mappings  $f : K \rightarrow A$ . Equip  $\mathcal{C}(K, A)$  with the norm and with the multiplication given by

$$\begin{aligned} \|f\| &= \sup\{\|f(t)\|; t \in K\}, \quad f \in \mathcal{C}(K, A), \\ (f \cdot g)(t) &= f(t) \cdot g(t), \quad t \in K, \quad f, g \in \mathcal{C}(K, A). \end{aligned}$$

- (1) Show that  $\mathcal{C}(K, A)$  is a Banach algebra.
- (2) Show that  $\mathcal{C}(K, A)$  is unital if and only if  $A$  is unital and find the unit.
- (3) Show that  $\mathcal{C}(K, A)$  is commutative if and only if  $A$  is commutative.

**Problem 10.** Let  $X$  be a Banach space. Consider the space  $L(X)$  of all continuous linear operators on  $X$  equipped with the operator norm. Define the multiplication on  $L(X)$  as composing the operators.

- (1) Show that  $L(X)$  is a unital Banach algebra and find its unit.
- (2) Show that  $L(X)$  is commutative if and only if  $\dim X = 1$ .

*Hint:* (2) If  $\dim X \geq 2$ , choose  $x_1, x_2 \in X$  linearly independent. Show that there exist  $x_1^*, x_2^* \in X^*$  such that  $x_1^*(x_1) = x_2^*(x_2) = 1$  and  $x_1^*(x_2) = x_2^*(x_1) = 0$ . Consider the operators of the form  $x \mapsto x_i^*(x)x_j$  and their linear combinations.

**Problem 11.** Let  $X$  be a Banach space and let  $A = K(X)$  is the space of compact linear operators on  $X$ , considered as a subspace of  $L(X)$ .

- (1) Show that  $A$  is a closed subalgebra of  $L(X)$ , and so it is a Banach algebra.
- (2) Show that  $A$  is unital if and only if  $\dim X < \infty$ .
- (3) Show that  $A$  is commutative if and only if  $\dim X = 1$ .
- (4) Assume that  $\dim X = \infty$ . Let  $B = \text{span}(A \cup \{I\})$  as a subalgebra of  $L(X)$ . Show that  $B$  is (algebraically) isomorphic to  $A^+$ , but not isometric.

*Hint: (3) Show that the operators used in Problem 10(2) are compact.*

**Problem 12.** Let  $(G, +)$  be a commutative group. Equip the Banach space  $\ell^1(G)$  with the multiplication  $*$  defined by

$$(f * g)(x) = \sum_{y \in G} f(y)g(x - y), \quad f, g \in \ell^1(G).$$

Show that  $\ell^1(G)$  is then a unital commutative Banach algebra and find its unit.

**Problem 13.** Let  $(G, \cdot)$  be a non-commutative group. Equip the Banach space  $\ell^1(G)$  with the multiplication  $*$  defined by

$$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x), \quad f, g \in \ell^1(G).$$

Show that  $\ell^1(G)$  is then a unital non-commutative Banach algebra and find its unit.

**Problem 14.** Let  $A = L^1(\mathbb{R}^n)$  where  $n \in \mathbb{N}$  (with the standard norm). Define the multiplication  $*$  on  $A$  as the convolution, i.e.,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y) dy, \quad x \in \mathbb{R}^n, f, g \in A.$$

- (1) Show that  $A$  is a commutative Banach algebra.
- (2) Show that  $A$  has no unit.

*Hint: (1) Use properties of the convolution. (2) Assume that  $g$  is a unit. Then for each  $r > 0$  we have  $g * \chi_{B(0,r)} = \chi_{B(0,r)}$ . Deduce that  $\chi_{B(0,r)}(x) = \int_{B(x,r)} g$  for almost all  $x$ , in particular  $\int_{B(x,r)} g = 1$  almost everywhere  $B(0,r)$ . For sufficiently small  $r$  derive a contradiction with  $g \in L^1(\mathbb{R}^n)$ .*

**Problem 15.** Let  $A = L^1([0, 1])$  (with the standard norm). Define the multiplication  $*$  on  $A$  as the convolution, i.e.,

$$(f * g)(x) = \int_0^1 f(y)g(x - y \pmod{1}) dy, \quad x \in [0, 1], f, g \in A.$$

- (1) Show that  $A$  is a commutative Banach algebra.
- (2) Show that  $A$  has no unit

*Hint: (1) Use an analogous approach as in proving properties of the convolution. (2) Proceed similarly as in Problem 14(2).*

**Problem 16.** Let  $(G, +)$  be a commutative compact topological group. (I.e.,  $(G, +)$  is a commutative group equipped with a Hausdorff topology in which the operations  $(x, y) \mapsto x + y$  and  $x \mapsto -x$  are continuous, which is moreover compact in this topology.) Let  $\mathcal{M}(G)$  be the space of all the complex Radon measures on  $G$ , equipped with the total variation norm and with the multiplication  $*$  defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x + y \in A\}),$$

where  $\mu \times \nu$  denotes the respective product measure. Show that  $\mathcal{M}(G)$  is then a unital commutative Banach algebra and find its unit.

**Problem 17.** Let  $(G, \cdot)$  be a non-commutative compact topological group. (I.e.,  $(G, \cdot)$  is a non-commutative group equipped with a Hausdorff topology in which the operations  $(x, y) \mapsto x \cdot y$  and  $x \mapsto x^{-1}$  are continuous, which is moreover compact in this topology.) Let  $\mathcal{M}(G)$  be the space of all the complex Radon measures on  $G$ , equipped with the total variation norm and with the multiplication  $*$  defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x \cdot y \in A\}),$$

where  $\mu \times \nu$  denotes the respective product measure. Show that  $\mathcal{M}(G)$  is then a unital non-commutative Banach algebra and find its unit.

**Problem 18.** Show that in the matrix algebra  $M_n$  an element has a right inverse if and only if it has a left inverse.

**Problem 19.** Let  $A = L(\ell^2)$ . Define two operators  $S, T \in A$  by

$$S(x_1, x_2, \dots) = (x_2, x_3, \dots) \quad \text{and} \quad T(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

- (1) Show that  $S$  and  $T$  are not invertible.
- (2) Show that  $S$  has a right inverse and describe all its right inverses.
- (3) Show that  $T$  has a left inverse and describe all its left inverses.

**Problem 20.** Let  $G = (\mathbb{Z}_n, +)$  where  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$  equipped with the addition modulo  $n$ . Let  $A = \ell^1(G)$  be the Banach algebra described in Problem 12.

- (1) Represent  $A$  as a subalgebra of the matrix algebra  $M_n$  (with an appropriate norm).
- (2) For  $n = 2$  and  $n = 3$  explicitly characterize invertible elements in  $A$ .

*Hint:* (1) Embed  $A$  into  $L(A)$ .

**Problem 21.** Let  $A$  be a Banach algebra. Define on  $A$  a new multiplication  $\odot$  by

$$x \odot y = y \cdot x, \quad x, y \in A.$$

- (1) Show that  $A^{op} = (A, \odot)$  is a Banach algebra.
- (2) Show that  $A^{op}$  need not be (algebraically) isomorphic to  $A$ .
- (3) Let  $X$  be a reflexive Banach space. Show that  $L(X)^{op}$  is isometrically isomorphic to  $L(X^*)$ .
- (4) Let  $H$  be a Hilbert space. Show that  $L(H)^{op}$  is isometrically isomorphic to  $L(H)$ .

*Hint:* (2) Use Problem 4.

PROBLEMS TO SECTION IV.2 – SPECTRUM AND ITS PROPERTIES

**Problem 22.** Let  $A = \mathcal{C}(K)$  for a compact Hausdorff space  $K$  and let  $f \in A$ .

- (1) Show that  $\sigma(f) = f(K)$ .
- (2) Compute the resolvent function of  $f$ .

**Problem 23.** Let  $A = \mathcal{C}_0(T)$  for a noncompact locally compact space  $T$ .

- (1) Show that  $\sigma(f) = f(T) \cup \{0\}$  for each  $f \in A$ .
- (2) Suppose that  $T$  is not  $\sigma$ -compact. Show that  $\sigma(f) = f(T)$  for each  $f \in A$ .
- (3) In case  $T = \mathbb{R}$  find an example of  $f \in A$  with  $f(T) \subsetneq \sigma(f)$ .

*Hint:* (1) Use the definition  $\sigma_A(f) = \sigma_{A^+}(f)$  and a description of  $A^+$  (for example that from Problem 8(3)). (2)  $\{t \in T; f(t) \neq 0\}$  is  $\sigma$ -compact.

**Problem 24.** Let  $A = \ell^1(\mathbb{Z}_n)$  (see Problem 20) and  $x \in A$ .

- (1) Characterize  $\sigma(x)$  as the set of eigenvalues of certain matrix.
- (2) For  $n = 2, 3$  compute  $\sigma(x)$  and the resolvent function explicitly.

*Hint:* Use the solution of Problem 20.

**Problem 25.** Let  $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ ,  $A = \mathcal{C}(\mathbb{T})$  and  $f(z) = z$  for  $z \in \mathbb{T}$ . Let  $B$  be the unital closed subalgebra of  $A$  generated by  $f$ , i.e.,

$$B = \overline{\text{span}}\{1, f, f^2, f^3, \dots\}.$$

Compute and compare  $\sigma_A(f)$  and  $\sigma_B(f)$ .

**Problem 26.** Let  $A$  be a unital Banach algebra and let  $x \in A$  be such that  $x^n = \mathbf{o}$  for some  $n \in \mathbb{N}$ . Determine  $\sigma(x)$  and compute the resolvent function.

**Problem 27.** Let  $A$  be a unital Banach algebra and let  $x \in A$  be such that  $x^2 = x$ . Determine  $\sigma(x)$  and compute the resolvent function.

*Hint:* Distinguish three cases:  $x = \mathbf{o}$ ,  $x = e$  and  $x \notin \{\mathbf{o}, e\}$ . The inverse of  $\lambda e - x$  find in the form  $\alpha e + \beta x$  for suitable  $\alpha, \beta \in \mathbb{C}$ .

**Problem 28.** Let  $A$  be a unital Banach algebra and let  $x \in A$  be such that  $x^3 = x$ . Determine  $\sigma(x)$  and compute the resolvent function.

*Hint:* There are several cases to be distinguished: The case  $x^2 = x$  is covered by Problem 27. The case  $x^2 = -x$  can be solved similarly as Problem 27. The next case to be solved is  $x^2 = e$ . Finally, if  $x^2 \notin \{e, x, -x\}$ , then show that  $e, x, x^2$  are linearly independent and find the inverse of  $\lambda e - x$  as a linear combination of  $e, x, x^2$ .

**Problem 29.** Let  $A = \ell^1(\mathbb{Z})$  (cf. Problem 12) and  $n \in \mathbb{Z}$ ,  $n \neq 0$ . Show that  $\sigma(\mathbf{e}_n) = \mathbb{T}$  (where  $\mathbf{e}_n$  is the respective canonical vector) and that

$$R(\lambda, \mathbf{e}_n) = \begin{cases} \sum_{k=0}^{\infty} \frac{\mathbf{e}_{kn}}{\lambda^{k+1}}, & |\lambda| > 1, \\ \sum_{k=1}^{\infty} -\lambda^k \mathbf{e}_{-kn}, & |\lambda| < 1. \end{cases}$$

*Hint:* This can be proved directly by solving the equation  $(\lambda \mathbf{e}_0 - \mathbf{e}_n) * f = \mathbf{e}_0$ . One can also use the formula from Proposition IV.8(v) and its modifications.

PROBLEMS TO SECTION IV.3 – HOLOMORPHIC FUNCTIONAL CALCULUS

**Problem 30.** Let  $A$  be a unital Banach algebra and let  $f$  be an entire function. Let

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad \lambda \in \mathbb{C},$$

be its Taylor expansion. Show that for each  $x \in A$  we have

$$\tilde{f}(x) = \sum_{n=0}^{\infty} a_n x^n.$$

*Hint:* Since  $f$  is an entire function, in the formula defining the holomorphic calculus we may integrate along a circle centered at zero with a sufficiently large radius.

**Problem 31.** Let  $A = M_n$  and  $D \in A$  be a diagonal matrix, with values  $d_1, \dots, d_n$  on the diagonal.

- (1) Show that  $\sigma(D) = \{d_1, \dots, d_n\}$  and compute the resolvent function.
- (2) Let  $f$  be a function holomorphic on a neighborhood of  $\sigma(D)$ . Show that  $\tilde{f}(D)$  is the diagonal matrix with values  $f(d_1), \dots, f(d_n)$  on the diagonal.
- (3) Deduce that in this case the value of  $\tilde{f}(D)$  depends only on  $f|_{\sigma(D)}$ .

*Hint:* (2) Consider a cycle consisting of sufficiently small circles with centers  $d_1, \dots, d_n$  and use the Cauchy formula for a disc.

**Problem 32.** Let  $A = M_n$  where  $n \geq 2$  and let  $J \in A$  be a Jordan cell, with the value  $z$  on the diagonal, i.e.,

$$J = \begin{pmatrix} z & 1 & 0 & \dots & 0 & 0 \\ 0 & z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z & 1 \\ 0 & 0 & 0 & \dots & 0 & z \end{pmatrix}.$$

- (1) Show that  $\sigma(J) = \{z\}$ .
- (2) Show that

$$(\lambda I - J)^{-1} = \begin{pmatrix} \frac{1}{\lambda-z} & \frac{1}{(\lambda-z)^2} & \dots & \frac{1}{(\lambda-z)^n} \\ 0 & \frac{1}{\lambda-z} & \dots & \frac{1}{(\lambda-z)^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda-z} \end{pmatrix} \quad \text{for } \lambda \in \mathbb{C} \setminus \{z\}.$$

- (3) Let  $f$  be a function holomorphic on a neighborhood of  $z$ . Show that

$$\tilde{f}(J) = \begin{pmatrix} f(z) & f'(z) & \frac{f''(z)}{2} & \dots & \frac{f^{(n-1)}(z)}{(n-1)!} \\ 0 & f(z) & f'(z) & \dots & \frac{f^{(n-2)}(z)}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f(z) \end{pmatrix}$$

- (4) Deduce that in this case the value of  $\tilde{f}(J)$  is not determined by  $f|_{\sigma(J)}$ .

*Hint:* (2) Consider a sufficiently small circle with center  $a$  and use the Cauchy formula for higher derivatives.

**Problem 33.** Let  $A = M_n$  and  $E \in A$  be an arbitrary matrix. Let  $f$  be a function holomorphic on a neighborhood of  $\sigma(E)$ .

- (1) Express  $\tilde{f}(E)$  using the Jordan canonical form of  $E$ .
- (2) Characterize those matrices  $E$  for which  $\tilde{f}(E)$  is determined by  $f|_{\sigma(E)}$ .

**Problem 34.** Let  $A = \ell^1(\mathbb{Z}_2)$  or  $A = \ell^1(\mathbb{Z}_3)$ . For  $x \in A$  and  $f$  holomorphic on a neighborhood of  $\sigma(x)$  compute the value of  $\tilde{f}(x)$ .

**Problem 35.** Let  $A$  be a unital Banach algebra and  $x \in A$  be an element satisfying one of the following conditions:

- (1)  $x^n = 0$  for some  $n \in \mathbb{N}$ ;
- (2)  $x^2 = x$ ;
- (3)  $x^2 = -x$ ;
- (4)  $x^2 = e$ ;
- (5)  $x^3 = x$ , but none of the conditions (2)–(4) holds.

Let  $f$  be a function holomorphic on a neighborhood of  $\sigma(x)$ . Compute  $\tilde{f}(x)$ . In which cases it is determined by  $f|_{\sigma(x)}$ ?

**Problem 36.** Let  $A = \mathcal{C}(K)$ , let  $g \in A$  and let  $F$  be a function holomorphic on a neighborhood of  $\sigma(g) = g(K)$ . Show that  $\tilde{F}(g) = F \circ g$ .

**Problem 37.** Let  $A = \ell^1(\mathbb{Z})$  (cf. Problem 12) and  $n \in \mathbb{Z}$ ,  $n \neq 0$ . By Problem 29 we know that  $\sigma(e_n) = \mathbb{T}$ . Let  $g$  be a function holomorphic on a neighborhood of  $\mathbb{T}$ . Show that

$$\tilde{g}(e_n) = \sum_{k \in \mathbb{Z}} a_k e_{kn},$$

where  $(a_k)_{k \in \mathbb{Z}}$  are the coefficients of the Laurent expansion of  $g$  in a neighborhood of  $\mathbb{T}$ .

*Hint:* One can use either the definitions and the formula from Problem 29, or one can prove an analogue of the statement in Problem 30 for Laurent series.

#### PROBLEMS TO SECTION IV.4 – IDEALS, MULTIPLICATIVE FUNCTIONALS AND THE GELFAND TRANSFORM

**Problem 38.** Let  $A = \mathcal{C}(K)$ .

- (1) Let  $I$  be a proper closed ideal in  $A$ . Show that there exists a nonempty closed set  $F \subset K$  such that

$$I = \{f \in \mathcal{C}(K); f|_F = 0\}.$$

- (2) Show that maximal ideals in  $A$  are exactly subspaces of the form

$$I = \{f \in \mathcal{C}(K); f(x) = 0\},$$

where  $x \in K$ .

- (3) Deduce that the multiplicative functionals on  $A$  are exactly the functionals of the form  $f \mapsto f(x)$ , where  $x \in K$ .
- (4) Explain and prove the statement, that the Gelfand transform of the algebra  $A$  is the identity mapping.

**Hint:** (1) Set  $F = \{x \in K; \forall f \in I : f(x) = 0\}$ . Show that  $F \neq \emptyset$  (otherwise using compactness and the definition of an ideal show that  $1 \in I$ ). Similarly show that for each closed set  $H$  disjoint with  $F$  there exists a nonnegative  $f \in I$  strictly positive on  $H$ . Using the Tietze theorem and definition of an ideal further show that there exists  $g \in I$  with values in  $[0, 1]$  which equals 1 on  $H$ . Finally deduce that any function which is zero on  $F$  may be approximated by functions from  $I$ .

**Problem 39.** Let  $A = M_n$ , where  $n \geq 2$ . Show that the only proper two-sided ideal in  $A$  is the zero ideal.

**Hint:** Assume that  $I$  is a nonzero ideal. Show that it contains at least one matrix with exactly one nonzero entry, then deduce that it contains all such matrices.

**Problem 40.** Let  $A = \ell^1(G)$ , where  $(G, +)$  is a commutative group (see Problem 12). Recall that then  $A^* = \ell^\infty(G)$ . Show that  $\varphi \in \ell^\infty(G)$  belongs to  $\Delta(A)$  if and only if it is a group homomorphism to the unit circle (i.e.,  $\varphi : G \rightarrow \mathbb{T}$  and  $\varphi(g_1 + g_2) = \varphi(g_1)\varphi(g_2)$  for  $g_1, g_2 \in G$ ).

**Hint:** Observe that  $e_{g_1} * e_{g_2} = e_{g_1+g_2}$ . To show that the values belong to  $\mathbb{T}$  use boundedness of  $\varphi$  and group operations.

**Problem 41.** Let  $A = \ell^1(\mathbb{Z})$ .

- (1) Describe  $\Delta(A)$  and explain how to understand the equality  $\Delta(A) = \mathbb{T}$ .
- (2) Describe the Gelfand transform of  $A$  and (using it) express the spectrum of a general element of  $A$ .
- (3) Is the Gelfand transform one-to-one? If yes, what is its inverse?
- (4) What is the range of the Gelfand transform? Is it onto?

**Hint:** (1) Use Problem 40 and consider the mapping  $\Delta(A) \ni \varphi \mapsto \varphi(1)$ . (3) Use the knowledge of Fourier series. (4) Not every continuous function has an absolutely convergent Fourier series.

**Problem 42.** Let  $A = \ell^1(\mathbb{Z}_n)$ , where  $n \in \mathbb{N}$ ,  $n \geq 2$ .

- (1) Describe  $\Delta(A)$  and show that it has exactly  $n$  elements.
- (2) Describe the Gelfand transform  $A$  and (using it) express the spectrum of a general element of  $A$ .
- (3) Is the Gelfand transform one-to-one? If yes, what is its inverse?
- (4) What is the range of the Gelfand transform? Is it onto?

**Hint:** (1) Use Problem 40 and consider the mapping  $\Delta(A) \ni \varphi \mapsto \varphi(1)$ . (3,4) Use (among others) properties of finite-dimensional spaces.

**Problem 43.** Let  $A = L^1(\mathbb{R}^n)$  (see Problem 14). Recall that  $A^* = L^\infty(\mathbb{R}^n)$ . Moreover,

$$\Delta(A) = \{\varphi : \mathbb{R}^n \rightarrow \mathbb{T}; \varphi \text{ is continuous and } \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \varphi(\mathbf{x} + \mathbf{y}) = \varphi(\mathbf{x}) \cdot \varphi(\mathbf{y})\},$$

which is a nontrivial known fact.

- (1) Show that the elements  $\Delta(A)$  are exactly the functions  $\mathbf{x} \mapsto e^{i\langle \mathbf{x}, \mathbf{y} \rangle}$ , where  $\mathbf{y} \in \mathbb{R}^n$ .
- (2) Explain what the equality  $\Delta(A) = \mathbb{R}^n$  means.
- (3) Describe the Gelfand transform of  $A$  and explain its relationship to the Fourier transform.



**Problem 44.** Consider  $\mathbb{T}$  as a compact group (the operation is the multiplication), i.e.,  $\mathbb{T} = \{e^{it}; t \in [0, 2\pi)\} = \{e^{it}; t \in \mathbb{R}\}$ . Let  $A = L^1(\mathbb{T})$  be equipped with the standard norm and the convolution as a multiplication, i.e.,

$$\|f\| = \frac{1}{\pi} \int_0^{2\pi} |f(e^{it})| dt, \quad (f * g)(e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{is})g(e^{i(t-s)}) ds.$$

- (1) Show that  $A$  is a commutative Banach algebra with no unit.
- (2) Using the representation  $A^* = L^\infty(\mathbb{T})$  and the known nontrivial fact that  $\Delta(A) = \{\varphi : \mathbb{T} \rightarrow \mathbb{C}; \varphi \text{ is continuous and } \forall x, y \in \mathbb{T} : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}$ , show that the elements  $\Delta(A)$  are exactly the functions  $x \mapsto x^n$  for  $n \in \mathbb{Z}$  (resp.  $e^{it} \mapsto e^{int}$ ).
- (3) Explain what the equality  $\Delta(A) = \mathbb{Z}$  means.
- (4) Describe the Gelfand transform of  $A$  and explain its relationship to the Fourier series.

*Hint:* (1) Show that it is just a different description of the algebra from Problem 15.

**Problem 45.** Let  $A$  be a unital commutative Banach algebra of finite dimension. Show that for each  $x \in A$  its spectrum  $\sigma(x)$  is a finite set with at most  $\dim A$  elements.

*Hint:* Let  $x \in A$ . It follows that there exists  $k \leq n$ , such that the element  $x^k$  is a linear combination of the elements  $1, x, \dots, x^{k-1}$ . Hence, for any  $\varphi \in \Delta(A)$  the value  $\varphi(x)$  must be a root of a certain polynomial of degree  $k$ . Moreover,  $\sigma(x) = \{\varphi(x); \varphi \in \Delta(A)\}$  (see Theorem IV.24(e)).

#### PROBLEMS TO SECTIONS IV.5 A IV.6 – $C^*$ -ALGEBRAS AND THE CONTINUOUS FUNCTION CALCULUS

**Problem 46.** Show that the Banach algebra  $\ell^1(\mathbb{Z})$  (see Problem 12) admits no involution making it a  $C^*$ -algebra.

*Hint:* Consider the Gelfand transform (see Problem 41) and show that its range is a proper dense subspace of  $\mathcal{C}(\mathbb{T})$ . Then use Theorem IV.33.

**Problem 47.** Show that the Banach algebra  $L^1(\mathbb{R})$  (see Problem 14) admits no involution making it a  $C^*$ -algebra.

*Hint:* Consider the Gelfand transform and its relationship to the Fourier transform (see Problem 43), use the known fact, that the Fourier transform is not onto  $\mathcal{C}_0(\mathbb{R}^n)$  and Theorem IV.33.

**Problem 48.** Let  $A = \mathcal{C}(K)$ , let  $g \in A$  and let  $F$  be a function continuous on  $\sigma(g) = g(K)$ . Show that  $\tilde{F}(g) = F \circ g$ .

- Problem 49.**
- (1) Let  $A$  be a commutative  $C^*$ -algebra and  $x \in A$ . Show that  $x$  is self-adjoint if and only if  $\sigma(x) \subset \mathbb{R}$ .
  - (2) Is this equivalence valid also for non-commutative  $C^*$ -algebras?

*Hint:* (1) For  $\Rightarrow$  use Proposition IV.32. For the converse note that the statement is valid in  $\mathcal{C}_0(T)$  and use Theorem IV.33. (2) Find a counterexample to  $\Leftarrow$  in the matrix algebra  $M_2$ .

**Problem 50.** Let  $A$  be a  $C^*$ -algebra. An element  $x \in A$  is called **positive** if it is self-adjoint and  $\sigma(x) \subset [0, +\infty)$ .

Show that each self-adjoint element may be expressed as a difference of two positive elements.

*Hint: Use the continuous function calculus for functions  $t \mapsto t^+$  and  $t \mapsto t^-$ .*

**Problem 51.** Let  $A$  be a unital  $C^*$ -algebra,  $x \in A$  a normal element and  $f \in \mathcal{C}(\sigma(x))$ .

- (1) Show that  $\tilde{f}(x)$  is a normal element.
- (2) Show that  $\tilde{f}(x)$  is a self-adjoint element if and only if the function  $f$  attains only real values.
- (3) Show that  $\tilde{f}(x)$  is a positive element if and only if the function  $f$  attains only non-negative values.
- (4) Show that  $\tilde{f}(x)$  is an invertible element if and only if the function  $f$  does not attain the value 0.

**Problem 52.** Let  $A$  be a  $C^*$ -algebra and let  $x, y \in A$  be two positive elements which commute (i.e.,  $xy = yx$ ). Show that  $xy$  is a positive element as well.

*Hint: Let  $B$  be a closed subalgebra of  $A$  generated by  $x$  and  $y$ . Then  $B$  is a commutative  $C^*$ -algebra. Use Theorem IV.33 and the fact that the statement is valid in  $\mathcal{C}_0(\Omega)$ .*

**Problem 53.** Let  $A$  be a  $C^*$ -algebra with a unit  $e$  and let  $x \in A$  be a normal element. Denote by  $B$  the closed subalgebra of  $A$  generated by the elements  $x, x^*, e$  and by  $B_0$  the closed subalgebra of  $A$  generated by the elements  $x, x^*$ .

- (1) Show that  $B = B_0$  if and only if  $x$  is invertible in  $A$ .
- (2) Let  $f \in \mathcal{C}(\sigma(x))$ . Show that  $\tilde{f}(x) \in B$ .
- (3) Let  $f \in \mathcal{C}(\sigma(x))$ . Show that  $\tilde{f}(x) \in B_0$  if and only if either  $x$  is invertible or  $f(0) = 0$ .

*Hint: (1) The implication  $\Leftarrow$  is obvious. If  $x$  is not invertible, deduce from Theorem IV.39 that  $B_0$  does not contain  $e$ . (2) Use Theorem IV.38. (3) The implication  $\Leftarrow$  follows from (1) and Theorem IV.39. Assume that  $x$  is not invertible (i.e.,  $0 \in \sigma(x)$ ) and  $f(0) \neq 0$ . Since the function calculus is an isomorphism of  $\mathcal{C}(\sigma(x))$  onto  $B$  and simultaneously an isomorphism of  $\mathcal{C}_0(\sigma(x) \setminus \{0\})$  onto  $B_0$ , it is enough to observe that  $f \notin \mathcal{C}_0(\sigma(x) \setminus \{0\})$ .*

**Problem 54.** Consider the situation from Problem 53 and, moreover, fix  $f \in C(\sigma(x))$ . Denote by  $D$  the closed subalgebra of  $A$  generated by the elements  $\tilde{f}(x), \tilde{f}(x)^*, e$  and by  $D_0$  the closed subalgebra of  $A$  generated by the elements  $\tilde{f}(x), \tilde{f}(x)^*$ .

- (1) Show that  $D \subset B$  and  $D_0 \subset B_0$ .
- (2) Show that  $D = D_0$  if and only if  $f$  does not attain the value zero.
- (3) Consider the diagram

$$\begin{array}{ccc} \Delta(B) & \xrightarrow{h_x} & \sigma(x) \\ r \downarrow & & \downarrow f \\ \Delta(D) & \xrightarrow{h_{\tilde{f}(x)}} & \sigma(\tilde{f}(x)) \end{array} ,$$

where  $r(\varphi) = \varphi|_D$  for  $\varphi \in \Delta(B)$  and  $h_x$  and  $h_{\tilde{f}(x)}$  are the mappings from the construction of the continuous calculus in Theorem IV.38, i.e.,  $h_x(\varphi) = \varphi(x)$  and  $h_{\tilde{f}(x)}(\psi) = \psi(\tilde{f}(x))$ . Show that this diagram commutes, i.e.,  $f \circ h_x = h_{\tilde{f}(x)} \circ r$ .

- (4) Deduce that for each  $g \in \mathcal{C}(\sigma(\tilde{f}(x)))$  we have  $\tilde{g}(\tilde{f}(x)) = \widetilde{g \circ f}(x)$ .

**Hint:** (2) Use Problem 53(1). (3) It is necessary to show that  $\varphi(\tilde{f}(x)) = f(\varphi(x))$ . To this end use the definition of the continuous function calculus and the definition of the Gelfand transform. (4) Use (3) and the definition of the continuous function calculus.

**Problem 55.** Let  $A$  be a  $C^*$ -algebra and let  $x \in A$  be a positive element.

- (1) Show that there exists a positive element  $y \in A$  such that  $y^2 = x$ .
- (2) Is such  $y$  unique?

**Hint:** (1) Use the continuous calculus for the function  $t \mapsto \sqrt{t}$  to the element  $x$ . (2) Let  $z$  be a positive element satisfying  $z^2 = x$  and let  $y$  be the element obtained by the way described in (1). Using Problem 54(4) applied to  $f(t) = t^2$  and  $g(t) = \sqrt{t}$  show that  $z = y$ .