Preliminaries: • $h \in \mathcal{D}(\mathbb{R}^d)$, $h \geq 0$, $\operatorname{spt} h \subset U(0,1)$, $\int_{\mathbb{R}^d} h = 1$ (Prop II.4)

• $h_j(x) = j^d h(jx)$, $x \in \mathbb{R}^d$.

Then $h_j \in \mathcal{D}(\mathbb{R}^d)$, $h_j \geq 0$, $\operatorname{spt} h_j \subset U(0, \frac{1}{j})$, $\int_{\mathbb{R}^d} h_j = 1$

(Theorem IV.6 (i))

Lemma VII.1: $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$

Proof: • For $\Omega = \mathbb{R}^d$ it is Corollary IV.7. The general proof is similar.

• Assume $p \in [1,\infty)$, $f \in L^p(\Omega)$ and $\varepsilon > 0$

(1) $\exists\ K \subset \Omega$ compact s.t. $\| f - f \cdot \chi_K \|_p < \frac{\varepsilon}{2}$

[$\exists\ K_n$ compact s.t. $K_n \nearrow \Omega$. Then $f \cdot \chi_{K_n} \to f$ in $L^p(\Omega)$ by Lebesgue dominated convergence theorem]

(2) $g_n := f \cdot \chi_K * h_n$. Then $g_n \in C^\infty(\mathbb{R}^d)$ [Prop. IV.3, note that $f\chi_K \in L^1_{loc}(\mathbb{R}^d)$ when defined by 0 outside Ω]

Moreover, $\operatorname{spt} g_n \subset K + \operatorname{spt} h_n \subset K + U(0, \frac{1}{n})$, so $g_n \in \mathcal{D}(\mathbb{R}^d)$;

moreover, if n is large enough ($\frac{1}{n} < \operatorname{dist}(K, \mathbb{R}^d \setminus \Omega)$) we have $\operatorname{spt} g_n \subset \Omega$, thus $g_n \in \mathcal{D}(\Omega)$.]

Finally, by Theorem IV.6 (ii) we have $g_n \to f\chi_K$ in $L^p(\mathbb{R}^d)$

thus there is n_0 s.t. for $n \geq n_0$ we have $\|g_n - f\chi_K\|_p < \frac{\varepsilon}{2}$

(3) conclusion: Let $n \geq n_0$ and $\frac{1}{n} < \operatorname{dist}(K, \mathbb{R}^d \setminus \Omega)$

Then $g_n \in \mathcal{D}(\Omega)$ and $\|g_n - f\|_p \leq \|g_n - f\chi_K\|_p + \|f\chi_K - f\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Lemma VII.2 $\Omega \subset \mathbb{R}^d$ open

① Let μ be a signed or complex regular Borel measure on Ω.
Assume $\forall \varphi \in \mathcal{D}(\Omega): \int_\Omega f \, d\mu = 0$. Then $\mu = 0$

<u>Proof:</u> Assume $\mu \neq 0$. Then there is $B \subset \Omega$ Borel set s.t. $\mu(B) \neq 0$
Fix $\varepsilon > 0$, $\varepsilon < \frac{1}{3}|\mu(B)|$.
μ regular $\Rightarrow \exists$ K compact, G open s.t. $K \subset B \subset G \subset \Omega$
and $|\mu|(G \setminus K) < \varepsilon$
Set $\delta := \operatorname{dist}(K, \mathbb{R}^d \setminus G)$ [if $G = \Omega = \mathbb{R}^d$, take $\delta := 1$]
$V := K + U(0, \frac{\delta}{2})$. Then V is open, bdd, $K \subset V \subset \overline{V} \subset G$
Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \frac{\delta}{4}$ and set $\varphi := \chi_V * h_n$
Then $\varphi \in C^\infty(\mathbb{R}^d)$ (Prop. IV.3)

$$\operatorname{spt} \varphi \subset \overline{V} + \operatorname{spt} h_n \subset \overline{V} + U(0, \tfrac{1}{n}) \subset K + \overline{U(0, \tfrac{\delta}{2})} + U(0, \tfrac{\delta}{4}) \subset$$
$$\subset K + U(0, \delta) \subset G \subset \Omega$$

$\Rightarrow$ $\operatorname{spt}\varphi$ is compact (by the first inclusion) contained in Ω, so $\varphi \in \mathcal{D}(\Omega)$
Moreover, $0 \le \varphi \le 1$ and for $x \in K$ we have

$$\varphi(x) = \int_{\mathbb{R}^d} \chi_V(x-y)\, h_n(y)\, dy \underset{\substack{\uparrow \\ \operatorname{spt} h_n \subset U(0,\frac{1}{n})}}{=} \int_{U(0,\frac{1}{n})} \chi_V(x-y) h_n(y)\, dy \underset{\substack{\uparrow \\ x-y \in x + U(0,\frac{1}{n}) \subset x + U(0,\frac{\delta}{4}) \subset V \\ (\text{as } x \in K)}}{=} \int_{U(0,\frac{1}{n})} h_n(y)\, dy = 1$$

Hence

$$\left| \int_\Omega \varphi \, d\mu \right| = \left| \int_G \varphi \, d\mu \right| = \left| \int_K \varphi \, d\mu + \int_{G \setminus K} \varphi \, d\mu \right| \ge \left| \int_K \varphi \, d\mu \right| - \int_{G \setminus K} |\varphi| \, d|\mu|$$

$$\underset{\substack{0 \le \varphi \le 1 \\ \varphi = 1 \text{ on } K}}{\ge} |\mu(K)| - \underbrace{|\mu|(G \setminus K)}_{< \varepsilon} \ge |\mu(B)| - \underbrace{|\mu(B \setminus K)|}_{\le |\mu|(B \setminus K) \le |\mu|(G \setminus K) < \varepsilon} - \varepsilon > |\mu(B)| - 2\varepsilon > 0$$

So, we found $\varphi \in \mathcal{D}(\Omega)$ s.t. $\int_\Omega f \, d\mu \neq 0$. This completes the proof.

② Let $f \in L^1_{loc}(\Omega)$, $\int_\Omega f\varphi = 0$ for each $\varphi \in \mathcal{D}(\Omega)$.

Then $f = 0$ a.e.

Proof: Assume f is not 0 a.e.

Then $\exists a \in \Omega$ $\exists r > 0$: $\overline{U(a,r)} \subset \Omega$ and f is not 0 a.e. on $U(a,r)$

For $B \subset U(a,r)$ Borel set $\mu(A) = \int_A f$. Then μ is a regular Borel measure on $U(a,r)$, $\mu \not\equiv 0$

[at least one of the sets $[\operatorname{Re} f < 0]$, $[\operatorname{Re} f > 0]$, $[\operatorname{Im} f > 0]$, $[\operatorname{Im} f < 0]$ has non zero measure]

So, by ① $\exists \varphi \in \mathcal{D}(U(a,r))$: $\int_{U(a,r)} \varphi \, d\mu \neq 0$

Then also $\varphi \in \mathcal{D}(\Omega)$

and $\int_\Omega f \cdot \varphi = \int_{U(a,r)} f \cdot \varphi = \int_{U(a,r)} \varphi \, d\mu \neq 0$