Proposition VIII.3 $f \in L^1_{loc}((a,b))$, constant zero function is a weak derivative of f $\Rightarrow$ f is constant ($\exists c \in \mathbb{F}: f = c$ a.e.)

Proof: Fix $\varphi_0 \in \mathcal{D}((a,b))$ with $\int_a^b \varphi_0 = 1$

Set $c = \int_a^b f\varphi_0$. Then $f = c$ a.e.:

Let $\varphi \in \mathcal{D}((a,b))$ be arbitrary.

Set $\psi = \varphi - \varphi_0 \cdot \int_a^b \varphi$. Then $\psi \in \mathcal{D}((a,b))$ and $\int_a^b \psi = 0$

$$\left[\int_a^b \psi = \int_a^b \left(\varphi - \varphi_0 \cdot \int_a^b \varphi\right) = \int_a^b \varphi - \underbrace{\int_a^b \varphi_0}_{=1} \cdot \int_a^b \varphi = 0 \right]$$

Define $\eta(t) = \int_a^t \psi$, $t \in (a,b)$. Then $\eta' = \psi$ and $\eta \in \mathcal{D}((a,b))$

$\left[\eta' = \psi \Rightarrow \eta \in C^\infty; \ \operatorname{spt} \eta \subset [\min \operatorname{spt} \psi, \max \operatorname{spt} \psi] \subset (a,b) \right.$ because $\psi \in \mathcal{D}((a,b))$ and $\left. \int_a^b \psi = 0 \right]$

0 is a weak derivative of $f \Rightarrow$

$$0 = -\int_a^b 0 \cdot \eta = \int_a^b f \cdot \eta' = \int_a^b f \cdot \psi = \int_a^b f \cdot \left(\varphi - \varphi_0 \cdot \int_a^b \varphi\right) = \int_a^b f\varphi - \underbrace{\int_a^b f\varphi_0}_{=c} \cdot \int_a^b \varphi$$

$$= \int_a^b (f-c)\cdot\varphi$$

Since φ was arbitrary, we get $\forall \varphi \in \mathcal{D}((a,b)): \int_a^b (f-c)\varphi = 0$,

hence $f - c = 0$ a.e. by Lemma 2

It means that $f = c$ a.e.