(a) $\Lambda \in \mathcal{D}'((a,b))$, $\Lambda' = 0 \Rightarrow \exists c \in \mathbb{F} : \Lambda = \Lambda_c$

Proof: Let us prove $\ker \Lambda_1 \subset \ker \Lambda$:

$\varphi \in \ker \Lambda_1 \Rightarrow \Lambda_1(\varphi) = 0 \Rightarrow \int_a^b \varphi = 0$

define $\psi(t) = \int_a^t \varphi$, $t \in (a,b)$. Then $\psi \in \mathcal{D}((a,b))$ and $\psi' = \varphi$

Thus $\Lambda(\varphi) = \Lambda(\psi') = -\Lambda'(\psi) = 0$, so $\varphi \in \ker \Lambda$

It now follows that $\exists c \in \mathbb{F} : \Lambda = c \cdot \Lambda_1$ (for example by lemma VI.3)

Then $\Lambda = \Lambda_c$.

(b1) $\Omega = \prod_{j=1}^{d} (a_j, b_j)$, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$ whenever $|\alpha| = 1 \Rightarrow \exists c \in \mathbb{F} : \Lambda = \Lambda_c$

Proof: By induction on d. The case $d=1$ is covered by (a).

Assume $d \geq 2$ and the statement holds for $d-1$

Notation: $\Omega' = \prod_{j=1}^{d-1} (a_j, b_j)$; $x \in \Omega \Rightarrow x = (x', x_d)$, where $x' \in \Omega'$, $x_d \in (a_d, b_d)$

$\alpha \in \mathbb{N}_0^d \Rightarrow \alpha = (\alpha', \alpha_d)$, $\alpha' \in \mathbb{N}_0^{d-1}$, $\alpha_d \in \mathbb{N}_0$

Let $\Lambda \in \mathcal{D}'(\Omega)$ be s.t. $D^\alpha \Lambda = 0$ whenever $|\alpha| = 1$.

It means: $\forall \varphi \in \mathcal{D}(\Omega)\ \forall j \in \{1..d\}: \Lambda\left(\frac{\partial \varphi}{\partial x_j}\right) = 0$

Claim Let $\psi \in \mathcal{D}(\Omega)$. Then $\exists \varphi \in \mathcal{D}(\Omega)$ s.t. $\frac{\partial \varphi}{\partial x_d} = \psi \Leftrightarrow \forall x' \in \Omega' : \underset{\overset{\shortparallel}{0}}{\int_{a_d}^{b_d} \psi(x', x_d)\, dx_d}$

$\ulcorner \Rightarrow$: $\int_{a_d}^{b_d} \psi(x', x_d)\, dx_d = \int_{a_d}^{b_d} \frac{\partial \varphi}{\partial x_d}(x', x_d)\, dx_d = \left[\varphi(x', x_d)\right]_{x_d = a_d}^{b_d} = 0$

$\Leftarrow$: Define $\varphi(x', x_d) = \int_{a_d}^{x_d} \psi(x', t)\, dt$ $\lrcorner$

Define $T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega')$ by

$$T\varphi(x') = \int_{a_d}^{b_d} \varphi(x', x_d)\, dx_d$$

$\ulcorner \varphi \in \mathcal{D}(\Omega) \Rightarrow T\varphi \in \mathcal{D}(\Omega')$: $T\varphi \in C^\infty$ by differentiating w.r.t. a parameter

$\operatorname{spt} T\varphi \subset$ projection of $\operatorname{spt}\varphi$ to Ω', it is compact $\lrcorner$

T is clearly linear and $\operatorname{Ker} T \subset \operatorname{Ker} \Lambda$

$\ulcorner T\varphi = 0 \overset{\text{Claim}}{\Rightarrow} \exists \psi \in \mathcal{D}(\Omega) : \frac{\partial \psi}{\partial x_d} = \varphi \Rightarrow \Lambda(\varphi) = 0 \lrcorner$

Fix $\eta \in \mathcal{D}((a_d, b_d))$ with $\int_{a_d}^{b_d} \eta = 1$.

For $\varphi \in \mathcal{D}(\Omega')$ define $(\varphi.\eta)(x', x_d) = \varphi(x') \cdot \eta(x_d)$

Then $\varphi.\eta \in \mathcal{D}(\Omega)$

Define $\Lambda'(\varphi) = \Lambda(\varphi.\eta)$, $\varphi \in \mathcal{D}(\Omega')$

$\Rightarrow \Lambda' \in \mathcal{D}'(\Omega')$ $\ulcorner \Lambda'$ linear .. clear

$K \subset \Omega'$ compact $\Rightarrow K \times \operatorname{spt}\eta \subset \Omega$ compact

$\varphi \in \mathcal{D}_K(\Omega') \Rightarrow \varphi.\eta \in \mathcal{D}_{K \times \operatorname{spt}\eta}(\Omega)$

$|\Lambda'(\varphi)| = |\Lambda(\varphi.\eta)| \underset{\exists N \exists C}{\le} C \cdot \|\varphi.\eta\|_N \le C \cdot \|\eta\|_N \cdot \|\varphi\|_N$

$D^\alpha(\varphi\eta) = D^{\alpha'}\varphi \cdot \frac{\partial \eta}{\partial x_d^{\alpha_d}}$ $\lrcorner$

Moreover $\forall \alpha'$ s.t. $|\alpha'| = 1$: $D^{\alpha'}\Lambda' = 0$

$\ulcorner \varphi \in \mathcal{D}(\Omega')$, $j = 1, \dots, d-1$: $\frac{\partial}{\partial x_j} \Lambda'(\varphi) = -\Lambda'\left(\frac{\partial \varphi}{\partial x_j}\right) = -\Lambda\left(\frac{\partial \varphi}{\partial x_j} \cdot \eta\right) =$

$= -\Lambda\left(\frac{\partial}{\partial x_j}(\varphi.\eta)\right) = 0$ $\lrcorner$

By the induction hypothesis $\exists c \in \mathbb{F}$ s.t. $\Lambda' = \Lambda_c$ (in $\mathcal{D}'(\Omega')$)

We claim that $\Lambda = \Lambda_c$ (in $\mathcal{D}'(\Omega)$)

$\ulcorner \varphi \in \mathcal{D}(\Omega) \Rightarrow \varphi - T\varphi.\eta \in \mathcal{D}(\Omega)$, moreover

$\varphi - T\varphi.\eta \in \operatorname{Ker} T \subset \operatorname{Ker} \Lambda$, so

$\Lambda(\varphi) = \Lambda(T\varphi.\eta) = \Lambda'(T\varphi) = \Lambda_c(T\varphi) = \int_{\Omega'} c \cdot T\varphi =$

$= \int_{\Omega'} c \cdot \left(\int_{a_d}^{b_d} \varphi(x', x_d)\, dx_d\right) dx' = \int_{\Omega} c \cdot \varphi = \Lambda_c(\varphi)$ $\lrcorner$

(b2) Ω open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$ whenever $|\alpha| = 1$

$\Rightarrow \exists c \in \mathbb{F}$ s.t. $\Lambda = \Lambda_c$

Proof: By (b1) we know that $\forall\, Q \subset \Omega$ cuboid (regularly placed)

$\exists\, c \in \mathbb{F}$ s.t. $\Lambda|_{\mathcal{D}(Q)} = \Lambda_c$ (on $\mathcal{D}'(Q)$)

Let us show c does not depend on Q:

Fix a cuboid $Q_0 \subset \Omega$ and $c \in \mathbb{F}$ s.t. $\Lambda|_{\mathcal{D}(Q_0)} = \Lambda_c$

Set $A = \{x \in \Omega\,;\, \exists\, Q \subset \Omega$ cuboid s.t. $x \in Q$ & $\Lambda|_{\mathcal{D}(Q)} = \Lambda_c\}$

Then
- $A \neq \emptyset$ (as $Q_0 \subset \Omega$)
- A is open ($x \in A \Rightarrow \exists\, Q$ s.t. $\dots \Rightarrow Q \subset A$)
- A is closed in Ω: $x \in \overline{A} \cap \Omega$

Find Q a cuboid s.t. $x \in Q \subset \Omega$

then $\exists\, d \in \mathbb{F}$ s.t. $\Lambda|_{\mathcal{D}(Q)} = \Lambda_d$

Further, $Q \cap A \neq \emptyset$, so take $y \in Q \cap A$

$\Rightarrow \exists\, Q_1$, a cuboid s.t. $y \in Q_1 \subset \Omega$ and $\Lambda|_{\mathcal{D}(Q_1)} = \Lambda_c$

Then $Q \cap Q_1 \neq \emptyset$, it is a cuboid

and $\Lambda_d = \Lambda|_{\mathcal{D}(Q \cap Q_1)} = \Lambda_c \Rightarrow d = c$

Ω connected $\Rightarrow A = \Omega$

Finally, we shall show that $\Lambda = \Lambda_c$ on $\mathcal{D}(\Omega)$:

$\varphi \in \mathcal{D}(\Omega) \Rightarrow \operatorname{spt}\varphi$ is a compact subset of Ω

$\forall x \in \mathcal{D}(\Omega)$ find $\delta_x > 0$ s.t. $Q_x = \prod_{j=1}^{d} (x_j - \delta_x, x_j + \delta_x) \subset \Omega$

and $\Lambda|_{\mathcal{D}(Q_x)} = \Lambda_c$

$E_x = \prod_{j=1}^{d} (x_j - \frac{\delta_x}{2}, x_j + \frac{\delta_x}{2}) \Rightarrow E_x, x \in \operatorname{spt}\varphi$ is an open cover of $\operatorname{spt}\varphi$

$\Rightarrow \exists\, x^1, \dots, x^n \in \operatorname{spt}\varphi : \operatorname{spt}\varphi \subset E_{x^1} \cup \dots \cup E_{x^n}$

Let (h_k) be a smoothing kernel. Set $\psi_j = \chi_{E_{x_j}} * h_{k_j}$, where $\frac{1}{k_j} < \frac{\delta_{x_j}}{2}$

Then $\psi_j \in \mathcal{D}(\Omega)$, $\operatorname{supp} \psi_j \subset Q_{x_j}$, $0 \le \psi_j \le 1$, $\psi_j|_{E_{x_j}} = 1$

$$\left.\begin{aligned} \varphi_1 &:= \varphi \cdot \psi_1 \\ \varphi_2 &:= (\varphi - \varphi_1)\cdot \psi_2 \\ \varphi_3 &:= (\varphi - \varphi_1 - \varphi_2)\cdot \psi_3 \\ &\vdots \\ \varphi_n &= (\varphi - \varphi_1 - \cdots - \varphi_{n-1})\cdot \psi_n \end{aligned}\right\} \Rightarrow \begin{aligned} &\varphi_j \in \mathcal{D}(\Omega),\ \operatorname{supp} \varphi_j \subset Q_j \\ &\varphi_1 + \cdots + \varphi_n = \varphi \end{aligned}$$

[$x \notin \operatorname{supp} \varphi \Rightarrow \varphi_1(x) = \varphi_2(x) = \cdots = \varphi_n(x) = 0$

$x \in \operatorname{supp} \varphi \Rightarrow \exists j : \psi_j(x) = 1$. Let j be the first one

then $\varphi_j(x) = \varphi(x) - \varphi_1(x) - \cdots - \varphi_{j-1}(x)$

and $\varphi_i(x) = 0$ for $i > j$]

Hence $\Lambda(\varphi) = \sum_{j=1}^{n} \lambda(\varphi_j) = \sum_{j=1}^{n} \int_{Q_j} c \cdot \varphi_j = \sum_{j=1}^{n} \Lambda_c(\varphi_j) = \Lambda_c(\varphi)$.]