$\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}'(\Omega)$

- $G \subset \Omega$ open. Λ vanishes on G if $\Lambda(\varphi)=0$ whenever $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{spt}\varphi \subset G$
- $\operatorname{spt}\Lambda = \Omega \setminus \bigcup \{G \subset \Omega \text{ open}; \Lambda \text{ vanishes on } G\}$
  $= \{x \in \Omega;\ \forall \varepsilon > 0\ \exists \varphi \in \mathcal{D}(\Omega): \operatorname{spt}\varphi \subset U(x,\varepsilon)\ \&\ \Lambda(\varphi) \neq 0\}$
- Λ has compact support if $\operatorname{spt}\Lambda$ is a compact subset of Ω

Proposition VII.12: $\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}'(\Omega)$

(a) $\Lambda = \Lambda_f$ for some $f \in L^1_{loc}(\Omega)$. Then
$$\operatorname{spt}\Lambda = \operatorname{spt} f := \{x \in \Omega;\ \forall \varepsilon > 0: \lambda^d(\{y \in U(x,\varepsilon) \cap \Omega;\ f(y) \neq 0\}) > 0\}$$

$\ulcorner$ $\subset$: $x \notin \operatorname{spt} f \Rightarrow \exists \varepsilon > 0$ s.t. $f = 0$ a.e. on $U(x,\varepsilon) \cap \Omega$
$\Rightarrow \Lambda = \Lambda_f$ vanishes on $U(x,\varepsilon) \cap \Omega \Rightarrow x \notin \operatorname{spt}\Lambda$

$\supset$: Assume $x \in \operatorname{spt} f$. Let $\varepsilon > 0$ be arbitrary. Then f is not a.e. zero on $U(x,\varepsilon) \cap \Omega$. By Lemma VII.2 we deduce that there is $\varphi \in \mathcal{D}(U(x,\varepsilon) \cap \Omega)$ s.t. $\int_{U(x,\varepsilon)\cap\Omega} f\varphi \neq 0$. Then $\varphi \in \mathcal{D}(\Omega)$ and $\Lambda_f(\varphi) \neq 0$.

We have verified that $x \in \operatorname{spt}\Lambda$ $\lrcorner$

Remark If f is cts, this support coincide with the standard one:
Set $G = \{x \in \Omega;\ f(x) \neq 0\}$. Then G is open (by continuity of f)
- $x \notin \overline{G} \Rightarrow \exists \varepsilon > 0: U(x,\varepsilon) \cap G = \emptyset$. As $\lambda^d(\emptyset) = 0$, $x \notin \operatorname{spt} f$
- $x \in \overline{G} \Rightarrow \forall \varepsilon > 0: U(x,\varepsilon) \cap G \neq \emptyset$. Any nonempty open set has strictly positive λ^d-measure, so $x \in \operatorname{spt} f$

(b) $\Lambda = \Lambda_\mu$ for a measure $\mu \Rightarrow \operatorname{spt}\Lambda = \operatorname{spt}\mu := \Omega \setminus \bigcup \{G \subset \Omega \text{ open};\ \forall B \subset G \text{ Borel}: \mu(B) = 0\}$

$\ulcorner$ $G \subset \Omega$ open. Then Λ_μ vanishes on $G \Leftrightarrow \mu|_G = 0$ (i.e. $\forall B \subset G$ Borel: $\mu(B) = 0$)

$\ulcorner$ $\Leftarrow$ obvious
$\Rightarrow$: Lemma VII.2 $\lrcorner\lrcorner$

$\lrcorner$

(c) $\varphi \in \mathcal{D}(\Omega)$; $\operatorname{spt}\varphi \cap \operatorname{spt}\Lambda = \emptyset \Rightarrow \Lambda(\varphi) = 0$
(i.e. Λ vanishes on $\Omega \setminus \operatorname{spt}\Lambda$)

$\ulcorner$ $\operatorname{spt}\varphi \cap \operatorname{spt}\Lambda = \emptyset \Rightarrow \operatorname{spt}\varphi \subset \bigcup \{G \subset \Omega$ open; Λ vanishes on $G\}$
$\operatorname{spt}\varphi$ is compact $\Rightarrow$ there is a finite subcover, i.e., there are
$G_1, \dots, G_n \subset \Omega$ open, Λ vanishes on G_j for $j = 1 \dots n$
s.t. $\operatorname{spt}\varphi \subset G_1 \cup \dots \cup G_n$.
We will be done if we show that Λ vanishes on $G_1 \cup \dots \cup G_n$.
To this end it is enough to use induction and the following claim:

Claim: $G_1, G_2 \subset \Omega$ open, Λ vanishes on G_1 and on $G_2 \Rightarrow \Lambda$ vanishes on $G_1 \cup G_2$

Proof: Let $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{spt}\varphi \subset G_1 \cup G_2$

- If $\operatorname{spt}\varphi \subset G_1$ or $\operatorname{spt}\varphi \subset G_2$, then $\Lambda(\varphi) = 0$
- Assume $\operatorname{spt}\varphi \not\subset G_1$ and $\operatorname{spt}\varphi \not\subset G_2$

Let $L := \operatorname{spt}\varphi \setminus G_2$. Then L is a nonempty compact subset of G_1
Fix $\delta > 0$ s.t. $3\delta < \operatorname{dist}(L, \mathbb{R}^d \setminus G_1)$
Let (h_k) be a smoothing kernel and fix $k \in \mathbb{N}$ s.t. $\frac{1}{k} < \delta$
Set $\psi := h_k * \chi_{L + B(0, 2\delta)}$
Then $\psi \in C^\infty(\mathbb{R}^d)$, $\operatorname{spt}\psi \subset L + B(0, 2\delta) + \operatorname{spt} h_k \subset L + B(0, 2\delta) + U(0, \frac{1}{k}) \subset G_1$
Moreover, $\psi = 1$ on $L + B(0, \delta)$
Set $\varphi_1 = \psi \cdot \varphi$ and $\varphi_2 = (1 - \psi) \cdot \varphi$
Then $\varphi_1 \in \mathcal{D}(\Omega)$, $\operatorname{spt}\varphi_1 \subset \operatorname{spt}\psi \subset G_1$, so $\Lambda(\varphi_1) = 0$
$\varphi_2 \in \mathcal{D}(\Omega)$, $\operatorname{spt}\varphi_2 \subset \overline{\operatorname{spt}\varphi \setminus (L + B(0, \delta))} \subset$
$\subset \operatorname{spt}\varphi \setminus (L + U(0, \delta)) \subset \operatorname{spt}\varphi \setminus L \subset G_2$,
so $\Lambda(\varphi_2) = 0$

Hence $\Lambda(\varphi) = \Lambda(\psi \cdot \varphi + (1 - \psi)\varphi) = \Lambda(\varphi_1) + \Lambda(\varphi_2) = 0$ $\square$

$\lrcorner$

(d) Λ has compact support $\Rightarrow \exists N \in \mathbb{N}_0 \ \exists C > 0 : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N$
for $\varphi \in \mathcal{D}(\Omega)$

$\ulcorner$ $\operatorname{spt}\Lambda$ is a compact subset of $\Omega \Rightarrow \exists \delta > 0$ s.t. $\operatorname{spt}\Lambda + B(0,3\delta) \subset \Omega$

Then $K := \operatorname{spt}\Lambda + B(0,3\delta)$ is a compact subset of Ω

Let (h_n) be a smoothing kernel, fix $k \in \mathbb{N}$ s.t. $\frac{1}{k} < \delta$

and set $\psi = h_k * \mathbb{1}_{\operatorname{spt}\Lambda + B(0,2\delta)}$. Then $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$\operatorname{spt}\psi \subset \operatorname{spt}\Lambda + B(0,2\delta) + U(0,\frac{1}{k}) \subset K$

$\psi = 1$ on $\operatorname{spt}\Lambda + B(0,\delta)$

For $\varphi \in \mathcal{D}(\Omega)$ we have

$$\Lambda(\varphi) = \Lambda(\varphi \cdot \psi) + \underbrace{\Lambda(\varphi \cdot (1-\psi))}_{=0 \text{ by (c)}} = \Lambda(\varphi \cdot \psi)$$

since $\varphi(1-\psi) = 0$ on $\operatorname{spt}\Lambda + B(0,\delta)$

hence $\operatorname{spt}\varphi(1-\psi) \cap \operatorname{spt}\Lambda = \emptyset$

Finally: let $N \in \mathbb{N}_0$ and $C > 0$ be such that

$$|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N \text{ for } \varphi \in \mathcal{D}_K(\Omega)$$

Then for each $\varphi \in \mathcal{D}(\Omega)$ we have

$$|\Lambda(\varphi)| = |\Lambda(\varphi \cdot \psi)| \underset{\varphi \cdot \psi \in \mathcal{D}_K(\Omega)}{\leq} C \cdot \|\varphi \cdot \psi\|_N \leq C \cdot 2^N \cdot \|\psi\|_N \cdot \|\varphi\|_N$$

see the proof of Prop. 8(d) $\lrcorner$

(e) $\operatorname{spt}\Lambda = \{p\} \Leftrightarrow \exists N \in \mathbb{N}_0$, $c_\alpha \in \mathbb{F}$ for $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$ s.t.

$$\Lambda = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \Lambda_{\delta_p} \qquad [\delta_p = \text{Dirac measure at } p]$$

$\ulcorner$ $\Leftarrow$: clear

$\Rightarrow$: WLOG $p = 0$. So, assume that $0 \in \Omega$ and $\operatorname{spt}\Lambda = \{0\}$

By (d) we find $N \in \mathbb{N}_0$ and $C > 0$ s.t. $|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N$ for $\varphi \in \mathcal{D}(\Omega)$

We will show that Λ is a linear combination of $D^\alpha \Lambda_{\delta_0}$, $|\alpha| \leq N$

Due to Lemma VI.3 it is equivalent to $\bigcap_{|\alpha| \leq N} \operatorname{Ker} D^\alpha \Lambda_{\delta_0} \subset \operatorname{Ker}\Lambda$

So, it is enough to show:

$$(\ast)\quad \varphi\in\mathcal{D}(\Omega),\ D^\alpha\varphi(0)=0 \text{ for } |\alpha|\le N \implies \Lambda(\varphi)=0$$

Fix $r\in(0,\frac12)$ s.t. $B(0,2r)\subset\Omega$ and find $\psi\in\mathcal{D}(\mathbb{R}^d)$ s.t.
$0\le\psi\le1$, $\operatorname{spt}\psi\subset U(0,2r)$, $\psi\equiv1$ on $B(0,r)$ (see Example IV.2(4))

For $m\in\mathbb{N}$ define $\psi_m(x)=\psi(mx)$, $x\in\mathbb{R}^d$

Then $\psi_m\in\mathcal{D}(\mathbb{R}^d)$, $\operatorname{spt}\psi_m\subset U(0,\frac{2}{m}r)$, $\psi_m\equiv1$ on $B(0,\frac{r}{m})$

Hence, for each $\varphi\in\mathcal{D}(\Omega)$ we have

$$|\Lambda(\varphi)| \underset{(c)}{\overset{\uparrow}{=}} |\Lambda(\varphi\cdot\psi_m)|\le C\cdot\|\varphi\cdot\psi_m\|_N$$

Let us estimate $\|\varphi\cdot\psi_m\|_N$: Fix α, $|\alpha|\le N$ and $x\in\Omega$:

$$|D^\alpha(\varphi\cdot\psi_m)(x)| = \Big|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\varphi(x)\cdot D^{\alpha-\beta}\psi_m(x)\Big| = \Big|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\varphi(x)\cdot\frac{1}{m^{|\alpha|-|\beta|}}D^{\alpha-\beta}\psi(mx)\Big|$$

$$\le\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{1}{m^{|\alpha|-|\beta|}}\|\psi\|_N\cdot|D^\beta\varphi(x)|$$

Since $\operatorname{spt}\varphi\psi_m\subset U(0,\frac{2}{m}r)$, we deduce

$$\|D^\alpha(\varphi\cdot\psi_m)\|_\infty\le\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{1}{m^{|\alpha|-|\beta|}}\|\psi\|_N\cdot\|D^\beta\varphi|_{U(0,\frac{2}{m}r)}\|_\infty$$

Fix now $\varphi\in\mathcal{D}(\Omega)$ s.t. $D^\alpha\varphi(0)=0$ for $|\alpha|\le N$

Let $\eta>0$ be arbitrary

Fix $m\in\mathbb{N}$ s.t. $\forall x\in U(0,\frac{2}{m}r)\ \forall\alpha, |\alpha|=N$: $|D^\alpha\varphi(x)|<\eta$

Claim: $x\in U(0,\frac{2}{m}r)$, $|\alpha|\le N \implies |D^\alpha\varphi(x)|\le\eta\cdot d^{N-|\alpha|}\cdot\|x\|^{N-|\alpha|}$

[Backwards induction on $|\alpha|$:

- $|\alpha|=N$ .... by the choice of m
- Assume it holds for $|\alpha|=k$ and fix α with $|\alpha|=k-1$

Then $|D^\alpha\varphi(x)|=|D^\alpha\varphi(x)-D^\alpha\varphi(0)|=\Big|\frac{\partial}{\partial t}\big(D^\alpha\varphi(tx)\big)\Big|_{t=s}\Big|=$

for some $s\in(0,1)$ by the mean value theorem

e_j is the canonical vector in $\mathbb{R}^d$

$$= \left| \sum_{j=1}^{d} \left(D^{\alpha+e_j} \varphi(sx) \right) \cdot x_j \right| \leq \sum_{j=1}^{d} \left| D^{\alpha+e_j} \varphi(sx) \right| \cdot |t_j|$$

$$\underset{\text{induction hypothesis}}{\leq} \sum_{j=1}^{d} \eta \cdot d^{N-\ell} \cdot \|x\|^{N-\ell} \cdot \underbrace{|t_j|}_{\leq \|t\|} \leq d \cdot \eta \cdot d^{N-\ell} \|t\|^{N-k+1}$$

$$= \eta\, d^{N-\ell+1} \|t\|^{N-\ell+1}$$

$\lrcorner$

It follows that (for the above choice of m)

$$\|D^\alpha(\varphi \cdot \psi_m)\|_\infty \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{m^{|\alpha|-|\beta|}} \|\psi\|_N \cdot \|D^\beta \varphi|_{U(0, \frac{2}{m} r)}\|_\infty$$

$$\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{m^{|\alpha|-|\beta|}} \|\psi\|_N \cdot \eta\, d^{N-|\beta|} \cdot \left(\frac{2}{m} r\right)^{N-|\beta|}$$

$$= \eta \cdot \|\psi\|_N \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \underbrace{d^{N-|\beta|}}_{\leq d^N} \underbrace{(2r)^{N-|\beta|}}_{\leq 1} \cdot \underbrace{\frac{1}{m^{N+|\alpha|-2|\beta|}}}_{\leq 1}$$

$$\leq \eta \cdot \|\psi\|_N\, d^N \cdot 2^N$$

So, $|\Lambda(\varphi)| \leq C \cdot \eta \cdot \|\psi\|_N\, d^N \cdot 2^N$

$\eta > 0$ arbitrary $\Rightarrow$ $\Lambda(\varphi) = 0$

This completes the proof.