Schwartz space:

$$\mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) ; \forall \alpha \in \mathbb{N}_0^d \ \forall N \in \mathbb{N}_0 \ x \mapsto (1+\|x\|^2)^N D^\alpha f \text{ is bdd on } \mathbb{R}^d \}$$

Norms on $\mathcal{S}(\mathbb{R}^d)$

$$p_N(f) = \max_{|\alpha| \le N} \| x \mapsto (1+\|x\|^2)^N D^\alpha f \|_\infty , \quad f \in \mathcal{S}(\mathbb{R}^d)$$

<u>Prop. VII. 17</u> (a) $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space, when equipped with norms $(p_N)_{N \in \mathbb{N}_0}$

$\ulcorner$ $(p_N)_{N \in \mathbb{N}_0}$ are norms on $\mathcal{S}(\mathbb{R}^d)$, $p_0 \le p_1 \le p_2 \le \dots$

$\Rightarrow$ $\mathcal{S}(\mathbb{R}^d)$ is a metrizable LCS (see Prop. V.21), let ρ be the metric provided by the quoted proposition.

Completeness: Assume $(f_k) \subset \mathcal{S}(\mathbb{R}^d)$ is ρ-Cauchy

$\overset{\text{Prop V.21}}{\Rightarrow}$ $\forall N$: (f_k) is p_N-Cauchy.

$\Rightarrow \forall N \ \forall \alpha, |\alpha| \le N$: $(x \mapsto (1+\|x\|^2)^N D^\alpha f_k(x))_k$ is $\|\cdot\|_\infty$-Cauchy

$\Rightarrow \forall N \ \forall \alpha, |\alpha| \le N \ \exists g_{N,\alpha}$: $(1+\|x\|^2)^N D^\alpha f_k(x) \rightrightarrows g_{N,\alpha}(x)$ on $\mathbb{R}^d$

Then $g_{N,\alpha}$ are bdd cts functions on $\mathbb{R}^d$ (unif. limits of bdd cts functions)

Since $0 \le \frac{1}{(1+\|x\|^2)^N} \le 1$, we deduce that

$$D^\alpha f_k(x) \rightrightarrows \frac{g_{N,\alpha}(x)}{(1+\|x\|^2)^N} \text{ on } \mathbb{R}^d \quad \text{whenever } |\alpha| \le N$$

It follows that $\dfrac{g_{N,\alpha}(x)}{(1+\|x\|^2)^N}$ does not depend on N.

Hence $\forall \alpha\ \exists\ h_\alpha$ bdd cts function on $\mathbb{R}^d$ s.t.

$$h_\alpha(x) = \frac{g_{N,\alpha}(x)}{(1+\|x\|^2)^N} \quad \text{for } N \geq |\alpha|$$

Since $D^\alpha f_n \rightrightarrows h_\alpha$ on $\mathbb{R}^d$, we deduce (by the theorem on uniform limits of derivatives) that $h_\alpha = D^\alpha h_0$ for each α.

thus $h_0 \in C^\infty(\mathbb{R}^d)$. Moreover, given $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d$, let $\tilde{N} := \max\{N, |\alpha|\}$

Then $|(1+\|x\|^2)^N D^\alpha h_0(x)| \leq |(1+\|x\|^2)^{\tilde{N}} D^\alpha h_0(x)| =$

$= |(1+\|x\|^2)^{\tilde{N}} h_\alpha(x)| = |g_{\tilde{N},\alpha}(x)|$, which is bdd

Thus $h_0 \in \mathcal{S}(\mathbb{R}^d)$.

Further, for each N, α, $|\alpha| \leq N$:

$(1+\|x\|^2)^N D^\alpha f_n(x) \rightrightarrows g_{N,\alpha}(x) = (1+\|x\|^2)^N D^\alpha h_0(x)$,

so $f_n \to h_0$ in p_N for each N.

By Prop V.21 we conclude $f_n \to h_0$ in $\mathcal{S}$. $\lrcorner$

(b) $\mathcal{D}(\mathbb{R}^d)$ is a dense subspace of $\mathcal{S}(\mathbb{R}^d)$

$\ulcorner$ • Clearly $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, so we need to prove the density.

• Let $f \in \mathcal{S}(\mathbb{R}^d)$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on $U(0,1)$.

Let $f_n(x) = f(x) \cdot \varphi(\frac{x}{n})$, $x \in \mathbb{R}^d$. Then $f_n \in \mathcal{D}(\mathbb{R}^d)$.

Moreover $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$:

Fix $\alpha \in \mathbb{N}_0^d$, $N \in \mathbb{N}_0$, $|\alpha| \le N$:

$$\left|(1+\|x\|^2)^N \left(D^\alpha f(x) - D^\alpha f_n(x)\right)\right| = \left|(1+\|x\|^2)^N D^\alpha\left(\left(1-\varphi\left(\tfrac{x}{n}\right)\right) f(x)\right)\right|$$

$$= \left|(1+\|x\|^2)^N \left(\left(1-\varphi\left(\tfrac{x}{n}\right)\right) D^\alpha f(x) + \sum_{0 \ne \beta \le \alpha} \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_d}{\beta_d}\cdot(-1)\cdot\frac{1}{n^{|\beta|}} D^\beta \varphi\left(\tfrac{x}{n}\right) D^{\alpha-\beta} f(x)\right)\right|$$

$$= \begin{cases} 0 & \|x\| \le n \\ \le \underbrace{\left(\left|1-\varphi\left(\tfrac{x}{n}\right)\right| + \sum_{0\ne\beta\le\alpha} \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_d}{\beta_d}\cdot\frac{1}{n^{|\alpha|}}\left|D^\beta\varphi\left(\tfrac{x}{n}\right)\right|\right)}_{\le\, 1 + 2^N \|\varphi\|_N \text{ which is a constant.}} \cdot \underbrace{\sup_{\substack{|\beta|\le N \\ \|x\|\ge n}} \left|(1+\|x\|^2)^N D^\beta f(x)\right|}_{(*)} & \|x\| > n \end{cases}$$

$$(*) = \sup_{\substack{|\beta|\le N \\ \|x\|\ge n}} \left|\frac{(1+\|x\|^2)^{N+1} D^\beta f(x)}{(1+\|x\|^2)}\right| \le \frac{p_{N+1}(f)}{(1+n^2)} \xrightarrow{n\to\infty} 0$$

We deduce $(1+\|x\|^2)^N D_\alpha f_n(x) \overset{n\to\infty}{\rightrightarrows} (1+\|x\|^2)^N D_\alpha f(x)$ on $\mathbb{R}^d$, i.e., $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$. ┘

(c) $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \Rightarrow \varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$.

┌ Assume $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$.

Then $\exists R > 0$ s.t. $\forall n$: $\operatorname{supp} \varphi_n \subset U(0,R)$

Then $$p_N(\varphi_n - \varphi) = \max_{|\alpha|\le N} \left\| x \mapsto (1+\|x\|^2)^N \left(D^\alpha \varphi_n(x) - D^\alpha \varphi(x)\right)\right\|_\infty \le$$

$$\le (1+R^2)^N \max_{|\alpha|\le N} \|D^\alpha\varphi_n - D^\alpha\varphi\|_\infty = (1+R^2)^N \|\varphi_n - \varphi\|_N \xrightarrow{n\to\infty} 0$$ ┘

Def: • A <u>tempered distribution</u> on $\mathbb{R}^d$ is a cts linear functional on $\mathbb{R}^d$
• $\mathcal{S}'(\mathbb{R}^d)$ = the space of tempered distributions on $\mathbb{R}^d$.

<u>Remark</u> (distributions vs tempered distributions)

• $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \Lambda|\mathcal{D}(\mathbb{R}^d) \in \mathcal{D}'(\mathbb{R}^d)$

$\ulcorner$ Assume $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$. Let $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^d)$, $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \xRightarrow{\text{Prop. VII.17 (c)}} \varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$
$\Rightarrow \Lambda(\varphi_n) \to \Lambda(\varphi) \lrcorner$

• $\Lambda_1, \Lambda_2 \in \mathcal{S}'(\mathbb{R}^d)$, $\Lambda_1|\mathcal{D}(\mathbb{R}^d) = \Lambda_2|\mathcal{D}(\mathbb{R}^d) \Rightarrow \Lambda_1 = \Lambda_2$

$\ulcorner$ by Proposition VII.17 (b) we know that $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d) \lrcorner$

• $\mathcal{S}'(\mathbb{R}^d) \subset\subset \mathcal{D}'(\mathbb{R}^d)$

$\ulcorner$ by the previous two remarks $\Lambda \mapsto \Lambda|_{\mathcal{D}(\mathbb{R}^d)}$ is a linear bijection of $\mathcal{S}'(\mathbb{R}^d)$ onto a subspace of $\mathcal{D}'(\mathbb{R}^d) \lrcorner$

• Let $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$. Then Λ is tempered $\Leftrightarrow$ Λ admits a cts extension to $\mathcal{S}(\mathbb{R}^d)$

$\ulcorner$ it is an interpretation of the previous remark $\lrcorner$

<u>Proposition VII.18</u>

(a) $\Lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{F}$ linear. Then Λ is a tempered distribution
$\Leftrightarrow \exists N \in \mathbb{N}_0\ \exists C > 0 : |\Lambda(f)| \leq C \cdot p_N(f),\ f \in \mathcal{S}(\mathbb{R}^d)$

$\ulcorner$ By Proposition V.14 $\lrcorner$

(b) Let $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$. Then Λ is tempered
$\Leftrightarrow \exists N \in \mathbb{N}_0\ \exists C > 0 : |\Lambda(\varphi)| \leq C \cdot p_N(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^d)$

$\Rightarrow$: Λ is tempered $\Rightarrow \exists \tilde{\Lambda} \in \mathcal{S}'(\mathbb{R}^d) : \Lambda = \tilde{\Lambda}_{|\mathcal{D}(\mathbb{R}^d)}$. Use (a).

$\Leftarrow$: Assume $\exists C, N$ s.t. the inequality holds

- By Prop. V.14 we deduce that Λ is cts on $\mathcal{D}(\mathbb{R}^d)$ if $\mathcal{D}(\mathbb{R}^d)$ is considered as a subspace of $\mathcal{S}(\mathbb{R}^d)$.
  So, it may be continuously extended to $\mathcal{S}(\mathbb{R}^d)$ (for example by the Hahn-Banach theorem V.31), so Λ is tempered

Def: $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d)$ if $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \Lambda_n(\varphi) \to \Lambda(\varphi)$
(i.e., if $\Lambda_n \xrightarrow{w^*} \Lambda$)

Thm 19 $(\Lambda_n) \subset \mathcal{S}'(\mathbb{R}^d)$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \lim\limits_{n\to\infty} \Lambda_n(\varphi)$ exists.

Then $\Lambda(\varphi) := \lim\limits_{n\to\infty} \Lambda_n(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, is a tempered distribution.

Proof: This follows from Theorem V.29