Lemma: The Fourier tranform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$

Proof: ① By Theorem IV.14 (3) we know that the Fourier transform is a linear bijection of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.
It remains to prove continuity.

② Let $m = \lfloor \frac{d}{2} \rfloor + 1$. Then $C := \int_{\mathbb{R}^d} \frac{1}{(\|x\|^2+1)^m} \, dm_d(x) < +\infty$

Moreover, for any $f \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\|\hat{f}\|_\infty \underset{\substack{\uparrow \\ \text{easy by definitions}}}{\leq} \|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)| \, dm_d(x) = \int_{\mathbb{R}^d} \frac{(\|x\|^2+1)^m |f(x)|}{(\|x\|^2+1)^m} \, dm_d(x) \leq C \cdot p_m(f)$$

Conclusion: $\|\hat{f}\|_\infty \leq C \, p_m(f)$, $f \in \mathcal{S}(\mathbb{R}^d)$

③ Fix $N \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$
For $f \in \mathcal{S}(\mathbb{R}^d)$:

$$(1+\|x\|^2)^N D^\alpha \hat{f}(x) \underset{\substack{\uparrow \\ \text{Corollary IV.9 (6)}}}{=} (1+\|x\|^2)^N \overbrace{\left(y \mapsto (-i)^{|\alpha|} y^\alpha f(y)\right)}^{\wedge} (x)$$

$$= (-i)^{|\alpha|} \overbrace{\left[y \mapsto \check{P}(D) (y^\alpha f(y))\right]}^{\wedge} (x) = (\#)$$

↑ Thm IV.11 (b), $P(x) = (1+\|x\|^2)^N = (1 + \sum_{j=1}^d x_j^2)^N$

$\check{P}(x) = P(-x) = P(-ix) = (1 - \sum_{j=1}^d x_j^2)^N$ .... a polynomial of degree $2N$
so $\check{P}(D)\varphi = \sum_{|\beta| \leq 2N} a_\beta D^\beta \varphi$ for some $a_\beta \in \mathbb{R}$ (uniquely determined by N) ✓

$$(\#) = (-i)^{|\alpha|} \overbrace{\left(y \mapsto \sum_{|\beta| \leq 2N} a_\beta D^\beta (y^\alpha f(y))\right)}^{\wedge} (x)$$

So, $$\|x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \overset{②}{\leq} C \cdot p_m\left(y \mapsto \sum_{\beta \leq 2N} a_\beta D^\beta (y^\alpha f(y))\right)$$

Lemma VII.21 $\Rightarrow$ $f \mapsto \left(y \mapsto \sum_{\beta \leq 2N} a_\beta D^\beta (y^\alpha f(y))\right)$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$,

So $\exists$ $M = M_{N,\alpha} > 0$, $n = n_{N,\alpha} \in \mathbb{N}_0$ s.t.

$$p_m \left(y \mapsto \sum_{\beta \leq 2N} a_\beta D^\beta (y^\alpha f(y)) \right) \leq M \, p_n(f).$$

Thus

$$\| x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x) \|_\infty \leq C \cdot M \cdot p_n(f)$$

4) It follows that $p_N(\hat{f}) \leq C \cdot \tilde{M} \cdot p_{\tilde{n}}(f)$, where $\tilde{M} = \max\{M_{N,\alpha};\ |\alpha| \leq N\}$
$\tilde{n} = \max\{n_{N,\alpha};\ |\alpha| \leq N\}$

So, $f \mapsto \hat{f}$ is continuous

5) The inverse is also continuous:

- either we can use open mapping theorem (Thm V.30)
- or the fact that the inverse is $f \mapsto \hat{\hat{\hat{f}}}$ (Thm IV.14 (5)).