Fourier transform of tempered distributions

Def: $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \hat{\Lambda}(\varphi) = \Lambda(\hat{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

Remarks: (1) $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \hat{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$

[$\varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \hat{\varphi} \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \hat{\Lambda}$ well defined

$\varphi \mapsto \hat{\varphi}$ linear $\Rightarrow \hat{\Lambda}$ linear

$\varphi \mapsto \hat{\varphi}$ cts (by Lemma VII.24) $\Rightarrow \hat{\Lambda}$ cts]

(2) Another view:

$\mathcal{F}: \varphi \mapsto \hat{\varphi}$ is a cts linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

Consider the dual operator $\mathcal{F}': \mathcal{S}(\mathbb{R}^d)^* \to \mathcal{S}(\mathbb{R}^d)^*$

(cf. Problem 10 to chapter VI)

Since $\mathcal{S}(\mathbb{R}^d)^* = \mathcal{S}'(\mathbb{R}^d)$, we have

$$\mathcal{F}'(\Lambda)(\varphi) = \Lambda(\mathcal{F}\varphi) = \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi)$$

Theorem (properties of the Fourier transform on $\mathcal{S}'$)

(a) The Fourier transform is a linear bijection of $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$ and $\hat{\hat{\Lambda}} = \check{\Lambda}$, $\hat{\hat{\hat{\hat{\Lambda}}}} = \Lambda$ for $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$

[• $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \hat{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$ by Remark 1 above

• $\Lambda \mapsto \hat{\Lambda}$ is clearly linear

• $\hat{\hat{\Lambda}}(\varphi) = \hat{\Lambda}(\hat{\varphi}) = \Lambda(\hat{\hat{\varphi}}) = \Lambda(\check{\varphi}) = \check{\Lambda}(\varphi)$

Thm IV.14(5)

• $\hat{\hat{\hat{\hat{\Lambda}}}}(\varphi) = \hat{\hat{\Lambda}}(\check{\varphi}) = \Lambda(\check{\check{\varphi}}) = \Lambda(\varphi)$

• Fourier transform is onto, as $\forall \Lambda \in \mathcal{S}'(\mathbb{R}^d)\ \exists U \in \mathcal{S}'(\mathbb{R}^d): \hat{U} = \Lambda$

Take $U = \hat{\hat{\hat{\Lambda}}}$

• Fourier transform is one-to one:

$\hat{\Lambda}_1 = \hat{\Lambda}_2 \Rightarrow \hat{\hat{\hat{\hat{\Lambda}}}}_1 = \hat{\hat{\hat{\hat{\Lambda}}}}_2 \Rightarrow \Lambda_1 = \Lambda_2$]

(b) $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d) \Rightarrow \hat{\Lambda}_n \to \hat{\Lambda}$ in $\mathcal{S}'(\mathbb{R}^d)$

$\ulcorner$ Assume $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d)$. Then for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\hat{\Lambda}_n(\varphi) = \Lambda_n(\hat{\varphi}) \to \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi) \lrcorner$$

Remarks (a) also follows from properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ using abstract properties of dual operators on LCS

- Any dual operator is weak*-to-weak* cts (Problem 10(3) to Chapter VI), so (b) follows from this //

(c) $f \in L^1(\mathbb{R}^d) \Rightarrow \hat{\Lambda}_f = \Lambda_{\hat{f}}$

$$\ulcorner \hat{\Lambda}_f(\varphi) = \Lambda_f(\hat{\varphi}) = \int_{\mathbb{R}^d} f\hat{\varphi} \overset{\text{Prop. IV.8(e)}}{=} \int_{\mathbb{R}^d} \hat{f}\varphi = \Lambda_{\hat{f}}(\varphi) \lrcorner$$

(d) $f \in L^2(\mathbb{R}^d) \Rightarrow \hat{\Lambda}_f = \Lambda_{P(f)}$, where $P(f)$ is the Plancherel transform.

$\ulcorner f_n := f \cdot \chi_{U(0,n)} \Rightarrow f_n \in L^1(\mathbb{R}^d),\ f_n \to f$ in $L^2(\mathbb{R}^d)$

Moreover, $P(f) = \lim_{n\to\infty} \hat{f}_n$, limit taken in $L^2(\mathbb{R}^d)$
(cf. Theorem IV.17)

Then $\hat{\Lambda}_f(\varphi) = \Lambda_f(\hat{\varphi}) = \int_{\mathbb{R}^d} f\hat{\varphi}\,d\lambda_d = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n\hat{\varphi}\,d\lambda_d = \lim_{n\to\infty} \int_{\mathbb{R}^d} \hat{f}_n\varphi\,d\lambda_d$

$f_n \to f$ in $L^2(\mathbb{R}^d)$, $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ — Prop. IV.8(e)

$\hat{f}_n \to P(f)$ in $L^2(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$

$$= \int_{\mathbb{R}^d} P(f)\varphi\,d\lambda_d = \Lambda_{P(f)}(\varphi) \lrcorner$$

(e) P polynomial on $\mathbb{R}^d \Rightarrow \widehat{P(D)\Lambda} = \check{P}\cdot\hat{\Lambda}$, $\widehat{P.\Lambda} = \check{P}(D)\hat{\Lambda}$

Recall: $\check{P}(t) = P(it)$, $P(D)f = \sum_{|\alpha|\le N} c_\alpha D^\alpha f$ if $P(x) = \sum_{|\alpha|\le N} c_\alpha x^\alpha$

$$\widehat{P(D)\Lambda}(\varphi) = P(D)\Lambda(\hat{\varphi}) = \Lambda(\check{P}(D)\hat{\varphi}) = \Lambda(\widehat{\check{P}.\varphi}) =$$

definition of derivative of Λ see Prop VII.22 (a)

Thm IV.11 (b) + $\check{\check{P}} = \check{P}$

$$= \hat{\Lambda}(\check{P}.\varphi) = \check{P}.\hat{\Lambda}(\varphi)$$

$$\widehat{P.\Lambda}(\varphi) = P\Lambda(\hat{\varphi}) = \Lambda(P\hat{\varphi}) = \Lambda(\widehat{\check{P}(D)\varphi}) = \hat{\Lambda}(\check{P}(D)\varphi)$$

Prop. VII.22 (b)

Thm IV.11 (b) + $\check{\check{P}} = P$

Prop. VII.22 (a)

$$= \check{P}(D)\hat{\Lambda}(\varphi)$$

Concrete examples:

$$\widehat{D^\alpha\Lambda} = (x\mapsto i^{|\alpha|}x^\alpha).\hat{\Lambda}$$

$$\widehat{(x\mapsto x^\alpha)\Lambda} = i^{|\alpha|}D^\alpha\hat{\Lambda}$$