Lemma VII.26 Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$

(a) $x_n \longrightarrow x$ in $\mathbb{R}^d \Rightarrow \tau_{x_n}\varphi \to \tau_x \varphi$ in $\mathcal{S}(\mathbb{R}^d)$

Γ • $x \in \mathbb{R}^d \Rightarrow \tau_x \varphi \in \mathcal{S}(\mathbb{R}^d)$

Γ see the proof of Prop. VII.22 (c) $\lrcorner\lrcorner$

• Fix $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le N$, $x, z \in \mathbb{R}^d$ and compute

$$\left|(1+\|y\|^2)^N D^\alpha \tau_z \varphi(y) - (1+\|y\|^2)^N D^\alpha \tau_x \varphi(y)\right| =$$

$$= (1+\|y\|^2)^N \left|D^\alpha \varphi(y-z) - D^\alpha \varphi(y-x)\right| = (1+\|y\|^2)^N \left|\int_0^1 \frac{d}{dt} D^\alpha \varphi(y-x-t(z-x))\,dt\right|$$

$$\le (1+\|y\|^2)^N \int_0^1 \left|\sum_{j=1}^d D^{\alpha+e_j}\varphi(y-x-t(z-x)) \cdot (x_j - z_j)\right| dt \le$$

$$\le (1+\|y\|^2)^N \int_0^1 \left(\sum_{j=1}^d |D^{\alpha+e_j}\varphi(y-x-t(z-x))|^2\right)^{1/2} \cdot \|x-z\|\,dt$$

$$\le \|x-z\| \cdot p_{N+1}(\varphi) \cdot (1+\|y\|^2)^N \cdot \int_0^1 \frac{\sqrt{d}}{(1+\|y-x-t(z-x)\|^2)^{N+1}}\,dt = (*)$$

Γ $1+\|y-x-t(z-x)\|^2 \ge 1 + (\|y-x\| - t\|z-x\|)^2 = 1+\|y-x\|^2 - 2t\|y-x\|\|z-x\|$

$+ t^2\|z-x\|^2 \ge 1 + \|y-x\|^2 - \|y-x\|$

if $\|z-x\| \le \frac{1}{2}$ $\lrcorner\lrcorner$

if $\|z-x\| \le \frac{1}{2}$

$$(*) \le \|x-z\|\, p_{N+1}(\varphi) \cdot \sqrt{d}\; \underbrace{\frac{(1+\|y\|^2)^N}{(1+\|y-x\|^2 - \|y-x\|)^{N+1}}}_{\substack{\text{a cts function on } \mathbb{R}^d \text{ with limit } 0 \text{ at } \infty, \\ \text{so bdd by some constant } M_N}}$$

So, if $x_n \to x$, then $\exists n_0\ \forall n \ge n_0\ \|x_n - x\| \le \frac{1}{2}$.

For $n \ge n_0$: $p_N(\tau_{x_n}\varphi - \tau_x\varphi) \le \|x-x_n\| \cdot p_{N+1}(\varphi) \cdot \sqrt{d} \cdot M_N \xrightarrow{n\to\infty} 0$ $\lrcorner$

(b) $e \in \mathbb{R}^d \Rightarrow \partial_e \varphi \in \mathcal{S}_{\mathbb{R}^d}$. Moreover, $\varphi_r \xrightarrow{r \to 0} \partial_e \varphi$ in $\mathcal{S}(\mathbb{R}^d)$

where $\varphi_r(x) = \dfrac{\varphi(x+re) - \varphi(x)}{r}$

- $\partial_e \varphi = \sum_{j=1}^{d} e_j \cdot \dfrac{\partial \varphi}{\partial x_j} \in \mathcal{S}(\mathbb{R}^d)$ (cf. the proof of Lemm. VII.13 (5))

- $N \in \mathbb{N}_0$, $|\alpha| \le N$

$$|(1+\|x\|^2)^N D^\alpha(\varphi_r - \partial_e \varphi)(x)| = (1+\|x\|^2)^N \cdot \left| \frac{1}{r}(D^\alpha \varphi(x+re) - D^\alpha \varphi(x)) - \partial_e D^\alpha \varphi(x) \right|$$

$$= (1+\|x\|^2)^N \left| \frac{1}{r} \int_0^r \frac{d}{dt} D^\alpha \varphi(x+te)\, dt - \partial_e D^\alpha \varphi(x) \right| =$$

$$= (1+\|x\|^2)^N \left| \frac{1}{r} \int_0^r \sum_{j=1}^{d} D^{\alpha+e_j} \varphi(x+te) \cdot e_j \, dt - \sum_{j=1}^{d} D^{\alpha+e_j} \varphi(x) e_j \right|$$

$$\le (1+\|x\|^2)^N \left| \frac{1}{r} \int_0^r \sum_{j=1}^{d} (D^{\alpha+e_j} \varphi(x+te) - D^{\alpha+e_j}\varphi(x)) \cdot e_j \, dt \right|$$

$$= (1+\|x\|^2)^N \left| \frac{1}{r} \int_0^r \sum_{j=1}^{d} \int_0^t \frac{d}{ds} D^{\alpha+e_j} \varphi(x+se)\, ds \cdot e_j \cdot dt \right| =$$

$$= (1+\|x\|^2)^N \left| \frac{1}{r} \int_0^r \int_0^t \sum_{j=1}^{d} \sum_{k=1}^{d} D^{\alpha+e_j+e_k} \varphi(x+se)\, e_k e_j \, ds\, dt \right|$$

$$\le (1+\|x\|^2)^N \cdot p_{N+2}(\varphi) \cdot \frac{1}{|r|} \int_{[0,r]} \int_{[0,t]} \sum_{j,k=1}^{d} \frac{1}{(1+\|x+se\|^2)^{N+2}} |e_k||e_j| \, ds\, dt$$

$$\le p_{N+2}(\varphi) \cdot \|e\|_1^2 \cdot \frac{1}{|r|} \int_{[0,r]} \int_{[0,t]} \frac{(1+\|x\|^2)^N}{(1+\|x+se\|^2)^{N+2}} \, ds\, dt$$

if $\|re\| < 2$ (cf. the proof of (a))

$$\le p_{N+2}(\varphi) \|e\|_1^2 \frac{1}{|r|} \int_{[0,r]} \int_{[0,t]} \underbrace{\frac{(1+\|x\|^2)^N}{(1+\|x\|^2 - \|x\|)^{N+2}}}_{\text{bdd on } \mathbb{R}^d,\ \le C} \, ds\, dt$$

$$\le C \cdot p_{N+2}(\varphi) \cdot \|e\|_1^2 \cdot \frac{1}{|r|} \underbrace{\int_{[0,r]} |t|\, dt}_{= \frac{r^2}{2}} = C \cdot p_{N+2}(\varphi) \|e\|_1^2 \cdot \frac{|r|}{2}$$

So, $p_N(\varphi_r - \partial_e \varphi) \overset{\text{if } \|re\| < 2}{\le} C \cdot p_{N+2}(\varphi) \|e\|_1^2 \cdot \dfrac{|r|}{2} \xrightarrow{r \to 0} 0$ $\square$

Proposition VII.27 $\quad \varphi \in \mathcal{S}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$

(a) $\Lambda \in \mathcal{S}'(\mathbb{R}^{d_1}) \Rightarrow \psi(y) = \Lambda(x \mapsto \varphi(x,y)),\ y \in \mathbb{R}^{d_2}$, belongs to $\mathcal{S}(\mathbb{R}^{d_2})$
and $D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0,\alpha)}\varphi(x,y))$ for $\alpha \in \mathbb{N}_0^{d_2}$

Γ For $y \in \mathbb{R}^{d_2}$ define $\varphi_y(x) = \varphi(x,y)$, $x \in \mathbb{R}^{d_1}$

① $\forall y \in \mathbb{R}^{d_2}$: $\varphi_y \in \mathcal{S}(\mathbb{R}^{d_1})$ and hence ψ is well defined

Γ $\alpha \in \mathbb{N}_0^{d_1} \Rightarrow D^\alpha \varphi_y(x) = D^{(\alpha,0)}\varphi(x,y)$. So $\varphi_y \in C^\infty(\mathbb{R}^{d_1})$

Moreover $\left|(1+\|x\|^2)^N D^\alpha \varphi_y(x)\right| \le (1+\|x\|^2+\|y\|^2)^N \left|D^{(\alpha,0)}\varphi(x,y)\right|$,
so it is bdd $\rfloor$

② $y_n \to y$ in $\mathbb{R}^{d_2} \Rightarrow \varphi_{y_n} \to \varphi_y$ in $\mathcal{S}(\mathbb{R}^{d_1})$

Γ $y,z \in \mathbb{R}^{d_2}$, $\alpha \in \mathbb{N}_0^{d_1}$, $|\alpha| \le N$ $(N \in \mathbb{N}_0)$:

$$\left|(1+\|x\|^2)^N (D^\alpha \varphi_z(x) - D^\alpha \varphi_y(x))\right| = \left|(1+\|x\|^2)^N \left(D^{(\alpha,0)}\varphi(x,z) - D^{(\alpha,0)}\varphi(x,y)\right)\right|$$

$$= (1+\|x\|^2)^N \left|\int_0^1 \frac{d}{dt} D^{(\alpha,0)}\varphi(x, y+t(z-y))\,dt\right| = (1+\|x\|^2)^N \left|\int_0^1 \sum_{j=1}^{d_2} D^{(\alpha,e_j)}\varphi(x,y+t(z-y))\cdot(z_j-y_j)\,dt\right|$$

$$\le (1+\|x\|^2)^N \int_0^1 \Big(\sum_{j=1}^{d_2} \underbrace{\left|D^{(\alpha,e_j)}\varphi(x,y+t(z-y))\right|^2}_{\le \frac{p_{N+1}(\varphi)}{(1+\|x\|^2+\|y+t(z-y)\|^2)^{N+1}} \le \frac{p_{N+1}(\varphi)}{(1+\|x\|^2)^{N+1}}}\Big)^{1/2} \cdot \|z-y\|\,dt$$

$$\le (1+\|x\|^2)^N \cdot \sqrt{d_2} \cdot \frac{p_{N+1}(\varphi)}{(1+\|x\|^2)^{N+1}} \cdot \|z-y\| = \frac{\sqrt{d_2}\, p_{N+1}(\varphi)}{1+\|x\|^2} \|z-y\|$$

$$\le \sqrt{d_2}\, p_{N+1}(\varphi) \cdot \|z-y\|$$

$\Rightarrow p_N(\varphi_y - \varphi_z) \le \sqrt{d_2}\, p_{N+1}(\varphi) \cdot \|z-y\| \to 0$ for $y \to z$ $\rfloor$

③ ψ is cts — Γ use ② $\rfloor$

④ $\frac{\partial \psi}{\partial y_j}(y) = \Lambda\left(x \mapsto \frac{\partial \varphi}{\partial y_j}(x,y)\right)$

Γ $\frac{\partial \psi}{\partial y_j}(y) = \lim_{t\to 0} \frac{\psi(y+te_j)-\psi(y)}{t} = \lim_{t\to 0} \Lambda\left(x \mapsto \underbrace{\frac{\varphi(x,y+te_j)-\varphi(x,y)}{t}}_{\substack{\varphi_t(x,y) \text{ for } e=(0,e_j) \\ \text{using notation from L VIII.26 (5)}}}\right) = (*)$

L VIII.26 (5) $\Rightarrow$ $\varphi_t \to \partial_{(0,e_j)}\varphi = \frac{\partial}{\partial y_j}\varphi$ on $\mathcal{S}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$

then clearly $(\varphi_t)_y \to \left(\frac{\partial \varphi}{\partial y_j}\right)_y$ in $\mathcal{S}(\mathbb{R}^{d_1})$ $(\forall y \in \mathbb{R}^{d_2})$

so

$(*) = \Lambda\left(\left(\frac{\partial \varphi}{\partial y_j}\right)_y\right) = \Lambda\left(x \mapsto \frac{\partial \varphi}{\partial y_j}(x,y)\right)$ $\lrcorner$

⑤ $\psi \in C^\infty(\mathbb{R}^d)$, $D^\alpha \psi(y) = \Lambda\left(x \mapsto D^{(0,\alpha)}\varphi(x,y)\right)$

Γ, $|\alpha|=1$ ... formula from ④, continuity from ③ applied to $\frac{\partial \varphi}{\partial y_j}$.

- by induction: Assume it holds for $|\alpha|\le N$, let $|\alpha| = N+1$

$\Rightarrow \alpha = \beta + e_j$, $|\beta| = N$, $j\in\{1\ldots d_2\}$

$D^\alpha \psi(y) = D^{\beta+e_j}\psi(y) = \frac{\partial}{\partial y_j} D^\beta \psi(y) \underset{\substack{\uparrow \\ \text{induction hypothesis}}}{=} \frac{\partial}{\partial y_j} \Lambda\left(x\mapsto D^{(0,\beta)}\varphi(x,y)\right) =$

$\overset{④}{=} \Lambda\left(x \mapsto \frac{\partial}{\partial y_j} D^{(0,\beta)}\varphi(x,y)\right) = \Lambda\left(x\mapsto D^{(0,\alpha)}\varphi(x,y)\right)$

the continuity follows from ③ applied to $D^{(0,\alpha)}\varphi$ $\lrcorner$

⑥ $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$:

Γ • Prop. VIII.18 $\Rightarrow$ $\exists N_1 \exists C>0$: $|\Lambda(\eta)| \le C\cdot p_{N_1}(\eta)$, $\eta\in\mathcal{S}(\mathbb{R}^{d_1})$

• Fix $N\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^{d_2}$, $|\alpha|\le N$

$(1+\|y\|^2)^N |D^\alpha\psi(y)| = (1+\|y\|^2)^N \left|\Lambda\left(x\mapsto D^{(0,\alpha)}\varphi(x,y)\right)\right|$

$= |\Lambda \,(x \mapsto (1+\|y\|^2)^N D^{(0,\alpha)} \varphi(x,y))| \leq$

$\leq C \cdot p_{N_1}(x \mapsto (1+\|y\|^2)^N D^{(0,\alpha)}\varphi(x,y)) \leq C \cdot p_{N_1+N}(\varphi)$

Γ $|\beta| \leq N_1$: $|(1+\|x\|^2)^{N_1} D^\beta ((1+\|y\|^2)^N D^{(0,\alpha)} \varphi(x,y))|$

$= (1+\|x\|^2)^{N_1}(1+\|y\|^2)^N |D^{(\beta,\alpha)}\varphi(x,y)| \leq p_{N_1+N}(\varphi)$]]

Lemma RT (representation of tempered distributions):

$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \iff \exists N \in \mathbb{N}_0, \exists \mu_\alpha, |\alpha| \leq N_0$ finite measures on $\mathbb{R}^d$ s.t.
(regular, Borel signed or complex)

$$\Lambda(\varphi) = \sum_{|\alpha|\leq N} \int_{\mathbb{R}^d} (1+\|x\|^2)^N D^\alpha \varphi(x)\, d\mu_\alpha(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

Proof: $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \exists N \in \mathbb{N}_0\ \exists C>0 \quad \Lambda(\varphi) \leq C \cdot p_N(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d)$
[by Prop. VII.18]

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ let $T\varphi = (x \mapsto (1+\|x\|^2)^N D^\alpha \varphi(x))_{|\alpha|\leq N}$

Then T maps $\mathcal{S}(\mathbb{R}^d)$ to $(C_0(\mathbb{R}^d))^{\{\alpha, |\alpha|\leq N\}}$

Γ $x \mapsto (1+\|x\|^2)^{N+1} D^\alpha \varphi(x)$ is a bdd cts function on $\mathbb{R}^d$

so $x \mapsto (1+\|x\|^2)^N D^\alpha\varphi(x)$ is a cts function with limit 0 at ∞ $\lrcorner$

Equip $(C_0(\mathbb{R}^d))^{\{\alpha, |\alpha|\leq N\}}$ with the „max norm"

$$\|(f_\alpha)_{|\alpha|\leq N}\| = \max_{|\alpha|\leq N} \|f_\alpha\|_\infty$$

$$Y = T(\mathcal{S}(\mathbb{R}^d)) \subset\subset (C_0(\mathbb{R}^d))^{\{\alpha, |\alpha|\leq N\}}$$

$\Lambda \circ T^{-1}$ is a cts linear functional on Y, $\|\Lambda\circ T^{-1}\| \leq C$

$\overset{H-B}{\Rightarrow} \exists L \in ((C_0(\mathbb{R}^d))^{\{\alpha, |\alpha|\leq N\}})^*$ extending $\Lambda \circ T^{-1}$

Riesz representation thm for $C_0(\mathbb{R}^d) \Rightarrow \exists\ \mu_\alpha,\ |\alpha| \le N$ finite regular Borel measures on $\mathbb{R}^d$ (signed or complex)

s.t. $L((f_\alpha)_{|\alpha|\le N}) = \sum_{|\alpha|\le N} \int_{\mathbb{R}^d} f_\alpha \, d\mu_\alpha$

Then $\Lambda(\varphi) = L(T\varphi) = \sum_{|\alpha|\le N} \int_{\mathbb{R}^d} (1+\|x\|^2)^N D^\alpha \varphi(x) \, d\mu_\alpha(x)$ $\lrcorner$

(b) $\Lambda_1 \in \mathcal{S}'(\mathbb{R}^{d_1})$, $\Lambda_2 \in \mathcal{S}'(\mathbb{R}^{d_2})$

$\Rightarrow \Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))) = \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x,y)))$

$\ulcorner$ (a) $\Rightarrow$ both expressions are well defined

Lemma RT ... Λ_1 ... N_1, μ_α, $|\alpha| \le N_1$

Λ_2 ... N_2, ν_α, $|\alpha| \le N_2$

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))) = \sum_{|\beta|\le N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} D^\beta \Lambda_1(x \mapsto \varphi(x,y)) \, d\nu_\beta(y) =$$

$$\stackrel{(a)}{=} \sum_{|\beta|\le N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} \Lambda_1(x \mapsto D^{(0,\beta)} \varphi(x,y)) \, d\nu_\beta(y) =$$

$$= \sum_{|\beta|\le N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} \sum_{|\alpha|\le N_1} \int_{\mathbb{R}^{d_1}} (1+\|x\|^2)^{N_1} D^{(\alpha,\beta)} \varphi(x,y) \, d\mu_\alpha(x) \, d\nu_\beta(y)$$

$$\stackrel{\text{FUBINI}}{=} \sum_{|\alpha|\le N_1} \sum_{|\beta|\le N_2} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} (1+\|x\|^2)^{N_1} (1+\|y\|^2)^{N_2} D^{(\alpha,\beta)} \varphi(x,y) \, d\nu_\beta(y) \, d\mu_\alpha(x)$$

$$= \sum_{|\alpha|\le N_1} \int_{\mathbb{R}^{d_1}} (1+\|x\|^2)^{N_1} \Lambda_2(y \mapsto D^{(\alpha,0)} \varphi(x,y)) \, d\mu_\alpha(x)$$

$$= \sum_{|\alpha|\le N_1} \int_{\mathbb{R}^{d_1}} (1+\|x\|^2)^{N_1} D^\alpha \Lambda_2(y \mapsto \varphi(x,y)) \, d\mu_\alpha(x) = \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x,y)))$$