Let $f : M \to X$ be weakly $\mathcal{A}$-measurable and let $f(M)$ be separable. Then f is strongly $\mathcal{A}$-measurable

Proof ① WLOG X is separable

② Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X
For each $n \in \mathbb{N}$ let $\varphi_n \in X^*$ satisfy $\|\varphi_n\| = 1$, $|\varphi_n(x_n)| = \|x_n\|$.
(it exists due to the dual formula for the norm)

③ $\forall x \in X$: $\|x\| = \sup_{n \in \mathbb{N}} |\varphi_n(x)|$.

⌈ $\geq$ clear, as $\|\varphi_n\| = 1$ for each $n \in \mathbb{N}$
$\leq$: this holds for a dense set (for each x_n),
hence it holds for each $x \in X$
(both sides are cts, in fact 1-Lipschitz, functions on X) ⌋

④ $\forall x \in X$ the function $t \mapsto \|f(t) - x\|$ is $\mathcal{A}$-measurable

⌈ $\|f(t) - x\| = \sup_{n \in \mathbb{N}} |\varphi_n(f(t) - x)| = \sup_{n \in \mathbb{N}} |\varphi_n \circ f(t) - \varphi_n(x)|$.

$t \mapsto |\varphi_n \circ f(t) - \varphi_n(x)|$ is $\mathcal{A}$-measurable due to weak $\mathcal{A}$-measurability of f.
Hence $t \mapsto \|f(t) - x\|$ is $\mathcal{A}$-measurable, being the supremum of a sequence of $\mathcal{A}$-meas. functions. ⌋

⑤ For $k \in \mathbb{N}$, $n \in \mathbb{N}$ set $A_n^k = f^{-1}(U(x_n, \frac{1}{k})) =$
$= \{t \in M ; \|f(t) - x_n\| < \frac{1}{k}\} \in \mathcal{A}$ by ④
Moreover, since $\bigcup_{n \in \mathbb{N}} U(x_n, \frac{1}{k}) = X$, we get $\bigcup_{n \in \mathbb{N}} A_n^k = M$

$B_n^k := A_n^k \setminus \bigcup_{j<n} A_j^k \Rightarrow B_n^k \in \mathcal{A}$, $(B_n^k)_{n=1}^{\infty}$ is a disjoint cover of M

Define $g_k(t) = x_n, \quad t \in B_n^k$

$\Rightarrow \|g_k(t) - f(t)\| < \frac{1}{k}, \ t \in M \quad \Rightarrow g_k \rightrightarrows f$ on M

Moreover,

$$g_k(t) = \lim_{n\to\infty} \underbrace{\sum_{j=1}^{n} x_j \cdot \chi_{B_j^k}(t)}_{\text{simple measurable}}.$$

$\Rightarrow g_k$ is strongly $\mathcal{A}$-measurable

By Lemma 2 f is strongly $\mathcal{A}$-measurable, too.