(a) $p \in [1,\infty) \Rightarrow$ simple integrable functions are dense in $L^P(\mu; X)$

<u>Proof:</u> Let $f \in L^P(\mu; X)$. Then f is strongly μ-measurable, so there is a sequence (u_n) of simple measurable functions s.t. $u_n \to f$ a.e.

Define $f_n : \Omega \to X$ by

$$f_n(\omega) = \begin{cases} u_n(\omega), & \text{if } \|f(\omega) - u_n(\omega)\| < 2\|f(\omega)\| \\ 0 & \text{otherwise} \end{cases}$$

Then:
- f_n is a simple function
- f_n is measurable
  (as $\{\omega;\ \|f(\omega) - u_n(t)\| < 2\|f(t)\|\} \in \Sigma$)
- $f_n \to f$ a.e.
  [Let $\omega \in \Omega$ be such that $u_n(\omega) \to f(\omega)$
  - $f(\omega) \neq 0 \Rightarrow \exists n_0\ \forall n \geq n_0\ \|f(\omega) - u_n(\omega)\| < 2\|f(\omega)\|$
    then for $n \geq n_0$ $f_n(\omega) = u_n(\omega) \to f(\omega)$
  - $f(\omega) = 0 \Rightarrow f_n(\omega) = 0$ for each $n \in \mathbb{N}$]
- $\|f(\omega) - f_n(\omega)\| \leq 2\|f(\omega)\|$ for all $\omega \in \Omega$
  [$f_n(\omega) = 0 \Rightarrow$ trivial, $f_n(\omega) \neq 0 \Rightarrow \|f(\omega) - u_n(\omega)\| < 2\|f(\omega)\|$ and $f_n(\omega) = u_n(\omega)$]
  $\Rightarrow \int \|f(\omega) - f_n(\omega)\|^P d\mu(\omega) \leq 2^P \int \|f(\omega)\|^P d\mu(\omega) < \infty$
  $\Rightarrow f - f_n \in L^P(\mu; X) \Rightarrow f_n \in L^P(\mu; X) \Rightarrow$
  f_n simple integrable
- $\|f(\omega) - f_n(\omega)\|^P \to 0$ a.e., $2\|f(\omega)\|^P$ is a majorant,
  so $\int \|f(\omega) - f_n(\omega)\|^P d\mu(\omega) \to 0$ by Lebesgue dom. conv. th.
  Hence $f_n \to f$ in $L^P(\mu; X)$

(5) If $p \in [1,\infty)$, $L^P(\mu)$ separable, X separable, then $L^P(\mu;X)$ is separable.

Proof Let $\{z_n, n \in \mathbb{N}\}$ be a dense subset of X
Let $\{h_n, n \in \mathbb{N}\}$ be a dense subset of $L^P(\mu;X)$

We will show that

$$A = \left\{ \sum_{j=1}^{k} z_{n_j} \cdot h_{m_j} \; ; \; n_1, \dots, n_k \in \mathbb{N}, m_1, \dots, m_k \in \mathbb{N} \right\}$$

is a dense subset of $L^P(\mu;X)$

① A is ctble, $A \subset L^P(\mu;X)$

$z \in X$, $h \in L^P(\mu) \Rightarrow z \cdot h \in L^P(\mu;X)$, $\|z \cdot h\|_p = \|z\| \cdot \|h\|_p$

[zh has separable range ($\subset$ span $\{z\}$)
zh is μ-measurable

$$\int_\Omega \|z \cdot h(\omega)\|^p d\mu(\omega) = \int_\Omega \|z\|^p \cdot |h(\omega)|^p d\mu(\omega) = \|z\|^p \cdot \|h\|_p^p$$]

② Let $f \in L^P(\mu;X)$ and $\varepsilon > 0$.

By (a) there is g_1, simple integrable,

such that $\|g_1 - f\|_p < \varepsilon/3$

Then $g_1 = \sum_{j=1}^{k} x_j \cdot \chi_{E_j}$, where $x_j \in X$
$E_j \in \Sigma$ pw disjoint
$\mu(E_j) < \infty$ for each j

(3) Find $n_1, \dots, n_k \in \mathbb{N}$ s.t. $\| z_{n_j} - t_j \|$ is so small

that $$\sum_{j=1}^{k} \| t_j - z_{n_j} \|^p \mu(E_j) < \left(\frac{\varepsilon}{3}\right)^p$$

and set

$$g_2 := \sum_{j=1}^{k} z_{n_j} \chi_{E_j}$$ Then $\| g_2 - g_1 \|_p < \frac{\varepsilon}{3}$

(4) Find $m_1, \dots, m_k \in \mathbb{N}$ s.t. $\| \chi_{E_j} - h_{m_j} \|_p$ is so small

that $$\sum_{j=1}^{k} \| z_{n_j} \| \, \| \chi_{E_j} - h_{m_j} \|_p < \frac{\varepsilon}{3}$$

Then $g_3 := \sum_{j=1}^{k} z_{n_j} h_{m_j} \in A$ and $\| g_3 - g_2 \|_p < \frac{\varepsilon}{3}$

To sum up: $\| g_3 - f \|_p \leq \varepsilon$