Lemma IX.2 Let X be a vector space, $\emptyset \neq A \subset X$ convex

(a) $x \in A \Rightarrow$ ($x \in \text{ext}\, A \Leftrightarrow \{x\}$ is a face of A)

[clear from definitions]

(b) $F_1 \subset A$ face of A, $F_2 \subset F_1$ face of F_1
$\Rightarrow F_2$ is a face of A

[clearly F_2 is nonempty and convex
$x, y \in A$ $\frac{x+y}{2} \in F_2 \Rightarrow x, y \in F_1$ (as $F_2 \subset F_1$ and F_1 is a face of A)

Hence $x, y \in F_2$ (as F_2 is a face of F_1)]

(c) X HLCS, A compact, A contains at least two points
$\Rightarrow \exists\ F \subsetneq A$ a face, closed in A

[$x, y \in A$, $x \neq y$. By H-B $\exists\ f \in X^*$ $f(x) \neq f(y)$

Since A is compact and f cts, f attains max on A

Let $F = \{a \in A;\ f(a) = \max f(A)\}$

Then • $\emptyset \neq F \subsetneq K$ (as $f(x) \neq f(y)$, f is not constant on A)

• F is closed and convex
[[as f is cts and linear]]

• $\frac{x+y}{2} \in F$, $x, y \in A$

$\Rightarrow$ $\max f(A) = f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y)) \leq$
$\leq \frac{1}{2}(\max f(A) + \max f(A)) = \max f(A)$

$\Rightarrow$ there are equalities, i.e. $x, y \in F$

So, F is a closed face.]

Theorem IX.3 (Krein-Milman)

X HLCS, $K \subset X$ compact convex $\Rightarrow K = \overline{\text{co}}\,\text{ext}\,K$

Pf: Suppose $K \neq \emptyset$. Denote by $\mathcal{F}$ the family of closed faces in K. If $\mathcal{R} \subset \mathcal{F}$ is linearly ordered by "$\subset$", then $\bigcap \mathcal{R} \in \mathcal{F}$ [the intersection is compact nonempty, convex, the second property of a face is clear]

Thus, by Zorn's lemma, there is a minimal $F \in \mathcal{F}$.

Lemma IX.2 (c,b) $\Rightarrow$ F is a singleton, i.e. $F = \{x\}$ for some $x \in K$
Lemma IX.2 (a) $\Rightarrow$ $x \in \text{ext}\,K$.

Thus $\text{ext}\,K \neq \emptyset$

If $K \setminus \overline{\text{co}}\,\text{ext}\,K \neq \emptyset$, fix $x \in K \setminus \overline{\text{co}}\,\text{ext}\,K$. By

H-B separation theorem $\exists f \in X^*$ $f(x) > \sup f(\text{ext}\,K)$

Let $F = \{y \in K;\ f(y) = \max f(K)\}$. The F is a closed face (cf. the proof of L IX.2(c)). By the first part, we know that $\text{ext}\,F \neq \emptyset$, thus there is $y \in \text{ext}\,F$.

Since $F \cap \text{ext}\,K = \emptyset$, we have $y \notin \text{ext}\,K$ } a contradiction.
By L IX.2(b, a) we deduce $y \in \text{ext}\,K$ }