PROOF OF PROPOSITION IX. 6 (MILMAN THEOREM)

X HLCS, $K \subset X$ compact convex, $A \subset K$, $K = \overline{\mathrm{co}}\, A$
$\Rightarrow \mathrm{ext}\, K \subset \overline{A}$

① If $A \subset K$ is such that $K = \mathrm{co}\, A$, then $\mathrm{ext}\, K \subset A$
[by the very definition of extreme points]

② Let U be an absolutely convex open nbhd of 0 in X.
Then there is a finite set $F \subset \overline{A}$ with $F + U \supset \overline{A}$
(using compactness of $\overline{A}$)

Then $K = \overline{\mathrm{co}\, A} \subset \overline{\mathrm{co}\,((F+U)\cap K)} =$

$$= \overline{\mathrm{co}\Big(\bigcup_{x\in F} (x+\overline{U})\cap K\Big)} = \mathrm{co}\Big(\bigcup_{x\in F} (x+\overline{U})\cap K\Big)$$

$(x+\overline{U})\cap K$, $x \in F$ are compact convex sets, they are finitely many, so the convex hull of their union is compact *)

Thus by ① we get $\mathrm{ext}\, K \subset \bigcup_{x\in F} (x+\overline{U})\cap K \subset \overline{A} + \overline{U}$

Since U is arbitrary, we get $\mathrm{ext}\, K \subset \overline{A}$ and we are done

[$x \notin \overline{A} \Rightarrow \exists\, U$ absolutely convex open nbhd of 0 s.t.
$(x+U)\cap \overline{A} = \emptyset$ then $(x + \frac{1}{2}\overline{U}) \cap \overline{A} = \emptyset$, so

$x \notin \overline{A} + \frac{1}{2}\overline{U}$]

*) $H_1, \dots, H_n \subset X$ compact convex sets $\Rightarrow$ $\mathrm{co}\,(H_1 \cup \dots \cup H_n)$ is compact
[$H_1 \times \dots \times H_n \times \{(t_1,\dots,t_n) \in [0,1]^n;\ t_1+\dots+t_n = 1\}$ is compact and the mapping
$(x_1,\dots,x_n, (t_1,\dots,t_n)) \mapsto t_1x_1 + \dots + t_nx_n$ is cts, its range is $\mathrm{co}\,(H_1\cup\dots\cup H_n)$]