Proposition IX.7 $X$ HLCS, $K \subset X$ compact convex set

(a) $\mu \in P(K) \Rightarrow \exists!\, x \in K$ $\forall f: K \to \mathbb{R}$ cts affine
$f(x) = \int f \, d\mu$

[ This $x$ is called the barycenter of $\mu$, denoted by $r(\mu)$ ]

① Uniqueness: $x \neq y \overset{H-B}{\Rightarrow} \exists f \in X^*$ $f(x) \neq f(y)$ and $f$ is cts affine

② $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$, where $\alpha_i \geq 0$ $\sum \alpha_i = 1$.

Then $r(\mu) = \sum \alpha_i x_i$

([ $f$ cts affine $\Rightarrow \int f d\mu = \sum \alpha_i f(x_i) = f(\sum \alpha_i x_i)$ ])

③ Finitely supported measures are $w^*$-dense in $P(K)$.

Finitely supported probabilities $= \operatorname{co}\{\delta_x, x \in K\}$

[Corollary II.36 (a)]

Suppose $\mu \in P(K) \setminus \overline{\operatorname{co}\{\delta_x, x \in K\}}^{w^*}$. By H-B separation

then there is $f \in C(K)$ (the dual of $(M(K), w^*)$ )

s.t. $\underset{\int f d\mu}{f(\mu)} > \sup\{\underset{\int f d\delta_x = f(x)}{f(\delta_x)}; x \in K\} = \sup f(K) = \max f(K)$

But $\int f d\mu \underset{\mu \geq 0}{\leq} \int \max f(K) d\mu \underset{\mu \text{ is a probability}}{=} \max f(K)$ ↯

④ Let $\mu \in P(K)$. By ③ there is a net $\mu_\alpha$ of finitely supported probabilities s.t. $\mu_\alpha \xrightarrow{w^*} \mu$

By ② we have $r(\mu_\alpha) \in K$. Let $x \in K$ be a cluster point of the net $(r(\mu_\alpha))$. Then $x = r(\mu)$

$\uparrow\!\uparrow$ $f$ cts affine

$$\int f\,d\mu \underset{\mu_\alpha \xrightarrow{w^*} \mu}{\uparrow}= \lim_\alpha \int f\,d\mu_\alpha = \lim_\alpha f(r(\mu_\alpha)) \underset{\uparrow}{=} f(x)$$

$f$ cts, $x$ is a cluster point of $(r(\mu_\alpha))$

(3) The mapping $\mu \mapsto r(\mu)$ is cts and affine

- Affine: $r(t\mu + (1-t)\nu) = t\,r(\mu) + (1-t)\,r(\nu)$

$f$ cts affine $\Rightarrow$

$$f(t\,r(\mu) + (1-t)\,r(\nu)) = t f(r(\mu)) + (1-t) f(r(\nu)) =$$
$$= t\int f\,d\mu + (1-t)\int f\,d\nu = \int f\,d(t\mu + (1-t)\nu)$$

- cts: Since $K$ is compact, the original topology on $K$ coincides with the weak topology $\sigma(X, X^*)$. So, it is enough to show that $\forall x^* \in X^*$ $\mu \mapsto x^*(r(\mu))$ is $w^*$-cts. (Prop. VI.1 (b))

But $x^*(r(\mu)) = \int x^*\,d\mu$, which is $w^*$-cts as $x^*|_K \in C(K)$.

Theorem IX. 8 $X$ HLCS, $K \subset X$ convex compact

$\forall x \in K \ \exists \mu \in P(K)$ $\mu(\overline{\mathrm{ext}\, K}) = 1$, $x = r(\mu)$

Proof: Consider the mapping $r : P(K) \to K$ provided by Prop. 22. It is continuous.

We wish to show that

$$r(\{\mu \in P(K) ; \mu(\overline{\mathrm{ext}\, K}) = 1\}) = K$$

By theorem IX.3 we know that the image is dense in $K$

Further

$\{\mu \in P(K) ; \mu(\overline{\mathrm{ext}\, K}) = 1\}$ is a closed subset of $P(K)$

[ $\mu(\overline{\mathrm{ext}\, K}) < 1 \Rightarrow \exists$ $F \subset K \setminus \overline{\mathrm{ext}\, K}$ compact s.t. $\mu(F) > 0$. Urysohn lemma yields $f : K \to [0,1]$ cts, $f|_F = 1$, $f|_{\overline{\mathrm{ext}\, K}} = 0$

Then $\int f \, d\mu > 0$ ... $\{\nu \in P(K) ; \int f \, d\nu > 0\}$ is a $w^*$-open set containing $\mu$ and disjoint with $\{\nu \in P(K) ; \mu(\overline{\mathrm{ext}\, K}) = 1\}$ ]

Thus the image is compact. So, the image is whole $K$