Let $(X, \mathcal{T})$ be a HLCS whose topology is generated by a sequence $(p_n)_{n\geq 1}^{\infty}$ of seminorms

(1) WLOG $p_1 \leq p_2 \leq p_3 \leq \dots$

⌈ $q_n(x) := \max\{p_1(x), \dots, p_n(x)\}$ is also a seminorm and the family (p_n) generates the same topology as (q_n) ⌋

(2) $$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x-y)\}$$

is a translation invariant metric on X

⌈ $\rho(x,x) = 0$ ... clear

$x \neq y \Rightarrow \exists n : p_n(x-y) > 0$ (as X is Hausdorff)

Hence $\rho(x,y) > 0$

$\rho(x,y) = \rho(y,x)$ ... clear, as $p_n(x-y) = p_n(y-x)$

$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

⌈ for each $n \in \mathbb{N}$: $p_n(x-z) \leq p_n(x-y) + p_n(y-z)$

and hence also

$\min\{1, p_n(x-z)\} \leq \min\{1, p_n(x-y)\} + \min\{1, p_n(y-z)\}$ ⌋

ρ translation invariant ... clear ⌋

(3) For each $n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\{x ; p_n(x) < \varepsilon\} \subset \{x ; \rho(x,0) < \varepsilon + 2^{-n}\}$$

⌈ $p_n(x) < \varepsilon \Rightarrow \forall k \leq n : p_k(x) \leq p_n(x) < \varepsilon$, so

$$\rho(x,0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, p_k(x)\} = \sum_{k=1}^{n} \frac{1}{2^k} \min\{1, p_k(x)\} +$$

$$+ \sum_{k=n+1}^{\infty} \frac{1}{2^k} \min\{1, p_k(x)\} < \sum_{k=1}^{n} \frac{1}{2^k} \cdot \varepsilon + \sum_{k=n+1}^{\infty} \frac{1}{2^k} < \varepsilon + \frac{1}{2^n}$$ ⌋

(4) $\forall \varepsilon \in (0,1)\ \forall n \in \mathbb{N}: \quad \{x;\ \rho(t,0) < \frac{\varepsilon}{2^n}\} \subset \{t;\ p_n(t) < \varepsilon\}$

$\ulcorner \rho(t,0) < \frac{\varepsilon}{2^n} \implies \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, p_k(t)\} < \frac{\varepsilon}{2^n}$

$\implies \frac{1}{2^n} \cdot \min\{1, p_n(t)\} < \frac{\varepsilon}{2^n}$

$\implies \min\{1, p_n(t)\} < \varepsilon$

Since $\varepsilon < 1$, it follows $p_n(t) < \varepsilon$ $\lrcorner$

(5) ρ generates the topology $\mathcal{T}$

$\ulcorner$ (3) $\implies \forall r > 0$ $\{t;\ \rho(t,0) < r\}$ is a neighborhood of 0

$\ulcorner r > 0$ ... Fix $\varepsilon > 0$ and $n \in \mathbb{N}$ s.t. $\varepsilon + \frac{1}{2^n} < r$

Then $\{t;\ \rho(t,0) < r\} \supset \{t;\ \rho(t,0) < \varepsilon + \frac{1}{2^n}\} \supset \{t;\ p_n(t) < \varepsilon\}$ $\lrcorner$

(4) $\implies \{x;\ \rho(t,0) < r\}$, $r > 0$, is a base of nbhds of 0.

Hence, the topology generated by ρ has the same nbhds of 0 as $\mathcal{T}$, so it coincides with $\mathcal{T}$. $\lrcorner$

(6) Let $(x_k) \subset X$, $x \in X$

$\rho(x_k, x) \to 0 \iff \forall n \in \mathbb{N}: p_n(x_k - x) \to 0$

$\ulcorner \Rightarrow$: Assume $\rho(x_k, x) \to 0$. Fix $n \in \mathbb{N}$

$\varepsilon \in (0,1)$ .... $\exists k_0 \ \forall k \geq k_0 : \rho(x_k, x) < \frac{\varepsilon}{2^n}$

$\overset{(4)}{\Rightarrow} \forall k \geq k_0: p_n(x_k - x) < \varepsilon$

$\Leftarrow$: Fix $\varepsilon > 0$. Find $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \frac{\varepsilon}{2}$

$\exists k_0 \ \forall k \geq k_0: p_n(x_k - x) < \frac{\varepsilon}{2}$

By (3) we have for $k \geq k_0$ $\rho(x_k, x) < \frac{\varepsilon}{2} + \frac{1}{2^n} < \varepsilon$

(Note: we use $\rho(x_k, x) = \rho(x_k - x, 0)$, as ρ is translation invariant.) $\lrcorner$

(7) Let $(x_k) \subset X$.

(x_k) is ρ-Cauchy $\iff \forall n: (x_k)$ is p_n-Cauchy

$\ulcorner$ The proof is the same as in (6)

only we work with $\rho(x_k, x_\ell)$ for $k, \ell \geq k_0$
or $p_n(x_k - x_\ell)$ for $k, \ell \geq k_0$) $\lrcorner$